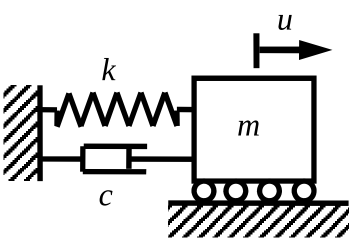


1. Eksplisiittinen Eulerin aikaintegroimiskaava on muotoa

$$x_{n+1} = x_n + \Delta t\, f_n$$

Esitä eksplisiittinen Eulerin integrointikaava kuvan värähtelijälle. Määritä integrointikaavan kriittinen aika-askel.

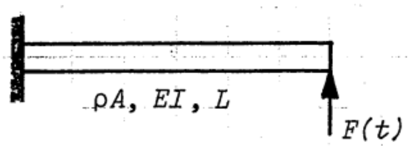
Derive explicit Euler integration scheme for the damped oscillator use above formula. Determinate the critical time step.

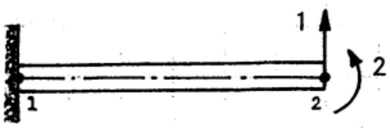
2. Kuvan ulokepalkilla on transienttikuormitus

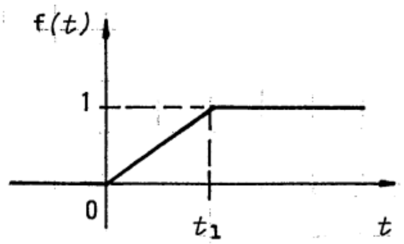
$$F(t) = F_0\, f(t)$$

missä $f(t)$ on oheisen kuvan mukainen ja $t_1 = \frac{1}{4}T_1$, T_1 on alin ominaisvärähdysaika. Määritä Newmarkin menetelmällä palkin ulokepään siirtymävaste ja piirrä käyrä aikavälillä $t \in [0, 2t_1]$. Käytä menetelmän parametreina $\beta = \frac{1}{4}, \gamma = \frac{1}{2}$.

For the cantilever beam the transient load is given above. Compute by Newmark's time stepping scheme the beam end displacement response when $t_1 = \frac{1}{4}T_1$ and T_1 is the lowest period of oscillation. Use parameters $\beta = \frac{1}{4}, \gamma = \frac{1}{2}$







Newmarkin integrointikaava/
Newmark's integration scheme:

1. Ennustajat/predictor:

$$\tilde{\mathbf{q}}_{n+1} = \mathbf{q}_n + \Delta t\, \dot{\mathbf{q}}_n + \frac{\Delta t^2}{2}\left(1 - 2\beta\right)\ddot{\mathbf{q}}_n$$

$$\tilde{\dot{\mathbf{q}}}_{n+1} = \dot{\mathbf{q}}_n + \Delta t\left(1 - \gamma\right)\ddot{\mathbf{q}}_n$$

2. Askel/step:

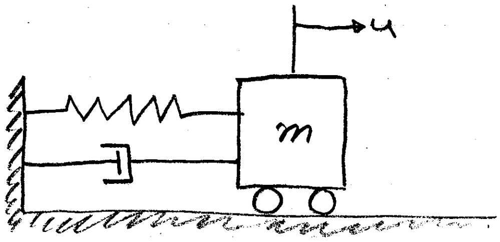
$$\left(\mathbf{M} + \Delta t\gamma\, \mathbf{C} + \Delta t^2 \beta \mathbf{K}\right)\ddot{\mathbf{q}}_{n+1} = \mathbf{F}_{n+1} - \mathbf{C}\tilde{\dot{\mathbf{q}}}_{n+1} - \mathbf{K}\tilde{\mathbf{q}}_{n+1}$$

ratkaise/solve $\ddot{\mathbf{q}}_{n+1}$

3. Korjaus/corrector:

$$\mathbf{q}_{n+1} = \tilde{\mathbf{q}}_{n+1} + \Delta t^2 \beta\, \ddot{\mathbf{q}}_{n+1}$$

$$\dot{\mathbf{q}}_{n+1} = \tilde{\dot{\mathbf{q}}}_n + \Delta t\, \gamma \ddot{\mathbf{q}}_{n+1}$$



$$\dot{x} = f(x, t)$$

$$x_{n+1} = x_n + F(t)$$

nyt $m\ddot{q} + c\dot{q} + kq = F(t)$, $\frac{k}{m} = \omega^2$, $\frac{c}{m} = 2\xi\omega$

1. Kertaluvun DY

$$\dot{q} = v$$

$$\dot{v} = \frac{F(t)}{m} - \frac{cv}{m} - \frac{kq}{m}$$

nyt $\underline{x} = \begin{bmatrix} q \\ v \end{bmatrix}$ $\quad \dot{\underline{x}} = \begin{bmatrix} v \\ \frac{F(t)}{m} - \frac{cv_n}{m} - \frac{kq_n}{m} \end{bmatrix}$

Euler eksplisiittinen

$$q_{n+1} = q_n + v_n \Delta t$$

$$v_{n+1} = v_n + \Big(\frac{F_n}{m} - \underbrace{\frac{cv_n}{m}}_{-2\xi\omega v_n} - \underbrace{\frac{kq_n}{m}}_{-\omega^2 q_n}\Big)\Delta t$$

$$\begin{bmatrix} q_{n+1} \\ v_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\omega^2 \Delta t & 1 - 2\xi\omega\Delta t \end{bmatrix}}_{[A]} \begin{bmatrix} q_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F_n}{m}\Delta t \end{bmatrix}$$

Lasketaan A:n ominaisarvot

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \quad (1-\lambda)(1 - 2\xi\omega\Delta t - \lambda) + \omega^2 \Delta t = 0$$

$$\lambda_{1,2} = 1 - \xi\omega\Delta t \pm \Delta t\,\omega \overbrace{\sqrt{\xi^2 - 1}}^{<0}$$

$$= 1 - \xi\omega\Delta t \pm i\,\Delta t\omega\sqrt{1-\xi^2}$$

$$|\lambda| = \sqrt{(1-\Delta t\xi\omega)^2 + \Delta t^2\omega^2(1-\xi^2)}$$

$$= \sqrt{1 - 2\Delta t\xi\omega + \cancel{\Delta t^2\xi^2\omega^2} + \Delta t^2\omega^2 \cancel{-\Delta t^2\omega^2\xi^2}}$$

$$= \sqrt{1 - 2\xi\Delta t\omega + \Delta t^2\omega^2}$$

$|\lambda|$ on vahvistuskertoimen amplitudi

$\Rightarrow$ vaatimus $|\lambda| \le 1$

ehto

$$|\lambda| = 1 \quad \Rightarrow \quad \sqrt{1 - 2\xi\omega\Delta t + \Delta t^2\omega^2} = 1$$

$$\cancel{1} - 2\xi\omega\Delta t + \Delta t^2\omega^2 = \cancel{1}$$

$$\Leftrightarrow \quad \omega\Delta t(\omega\Delta t - 2\xi) = 0$$

$$\Delta t_{kr} = \frac{2\xi}{\omega}$$

eli $\Delta t \le \Delta t_{kr}$

Huom. Jos $\xi = 0 \Rightarrow \Delta t_{kr} = 0$

/

1

$$x_{n+1} = x_n + F_n$$

Nyt $\quad m\ddot{q} + c\dot{q} + kq = F(t)$ $\qquad \frac{k}{m} = \omega^2$

$\frac{c}{m} = 2\zeta\omega$

1 kertaluvun DY

$$\dot{q} = v, \qquad m\dot{v} + cv + kq = F(t)$$
$$\dot{v} = F(t)/m - cv/m - kq/m$$

nyt $\underline{x} = \begin{pmatrix} q \\ v \end{pmatrix} \qquad \underline{f} = \begin{pmatrix} v \\ F(t)/m - cv/m - kq/m \end{pmatrix}$

Euler eksplisiittinen

$$q_{n+1} = q_n + \Delta t v_n$$
$$v_{n+1} = v_n + \Delta t (F_n/m \overbrace{- cv_n/m}^{-2\zeta\omega v_n} \overbrace{- kq_n/m}^{-\omega^2 q_n})$$

$$\begin{pmatrix} q_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\Delta t\omega^2 & 1-\Delta t 2\zeta\omega \end{bmatrix}}_{A} \begin{pmatrix} q_n \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta t F_n/m \end{pmatrix}$$

lasketaan A:n ominaisarvot

$$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(1-\Delta t 2\zeta\omega - \lambda) + \Delta t^2\omega^2 = 0$$

$$\lambda_{1,2} = 1 - \Delta t\zeta\omega \pm \Delta t\omega\sqrt{\overbrace{\zeta^2 - 1}^{<0}}$$
$$= 1 - \Delta t\zeta\omega \pm i\Delta t\omega\sqrt{1-\zeta^2}$$

$$|\lambda| = \sqrt{(1-\Delta t\zeta\omega)^2 + \Delta t^2\omega^2(1-\zeta^2)} = \sqrt{1 - 2\Delta t\zeta\omega + \cancel{\Delta t^2\zeta^2\omega^2} + \Delta t^2\omega^2 - \cancel{\Delta t^2\omega^2\zeta^2}}$$
$$= \sqrt{1 - 2\zeta\Delta t\omega + \Delta t^2\omega^2}$$

$|\lambda|$ on vahvistuskertoimen amplitudi

$\Rightarrow$ vaatimus $|\lambda| \leq 1$

ehto

$|\lambda| = 1 \Rightarrow = \sqrt{1 - 2\zeta\Delta t\omega + \Delta t^2\omega^2} = 1$

$\cancel{1} - 2\zeta\Delta t\omega + (\Delta t\omega)^2 = \cancel{1}$

$\Leftrightarrow \quad \Delta t\omega(\Delta t\omega - 2\zeta) = 0$

$$\boxed{\Delta t_{kr} = \frac{2\zeta}{\omega}}$$

Huom Jos $\zeta = 0 \Rightarrow \Delta t_{kr} = 0$

Tapa 3

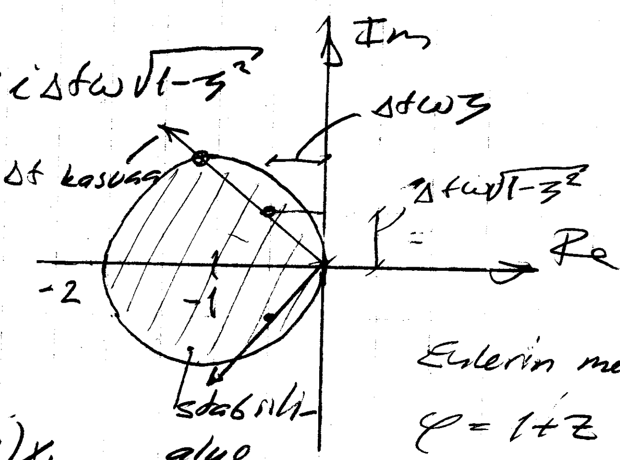
Jakobin matriisi $\qquad$ J:n ominaisarvot

$$J = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \Rightarrow \lambda_J = -\omega\zeta \pm i\omega\sqrt{1-\zeta^2}$$

$k/m = \omega^2 \qquad c/m = 2\zeta\omega$

$$J = \frac{\partial \underline{f}}{\partial x} \qquad \underline{f} = \begin{pmatrix} v \\ F(t)/m - cv/m - kq/m \end{pmatrix}$$

$z = \Delta t\lambda_J = -\Delta t\omega\zeta \pm i\Delta t\omega\sqrt{1-\zeta^2}$



testi ODY $\dot{x} = \lambda x$

$x_{n+1} = x_n + \Delta t f_n$

$= x_n + \Delta t\lambda x_n = (1 + \Delta t\lambda)x_n$

Eulerin menetelmä

$\varphi = 1 + z$

stabiilisuusehto

$|\varphi| \leq 1$

$\varphi(z) = 1 + z \qquad x_{n+1} = \varphi(z)x_n$

$\lambda = \lambda_J \Rightarrow \varphi(z) = 1 - \zeta z + i z\sqrt{1-\zeta^2} \quad Ok$

Harjoitus 15, tehtävä 2

Ominaiskulmataajuus $\omega_1 = 3.5327\sqrt{EI/\rho AL^4}$

jaksonaika $T_1 = 2\pi/\omega_1$

rampin aika $t_1 = T_1/4$

aika-askel $\Delta t = t_1/6$

matriisit $\mathbf{K} = \dfrac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}$, $\mathbf{M} = \dfrac{\rho AL}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}$

matriisi $\left(\mathbf{M} + \Delta t^2 \beta \mathbf{K}\right)^{-1} = \dfrac{1}{\rho AL^3}\begin{bmatrix} 6.98398 & 28.1945 \\ 28.1945 & 180.4187 \end{bmatrix}$

Alkuehdot:

$$\mathbf{q}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dot{\mathbf{q}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \ddot{\mathbf{q}}_0 = \mathbf{M}^{-1}\left(\mathbf{F}_0 - \mathbf{K}\mathbf{q}_0\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

1. askel $t = \Delta t$

voima $\mathbf{F}_1 = F_0\begin{pmatrix} 1/6 \\ 0 \end{pmatrix}$, Ennustajat: $\tilde{\mathbf{q}}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\tilde{\dot{\mathbf{q}}}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

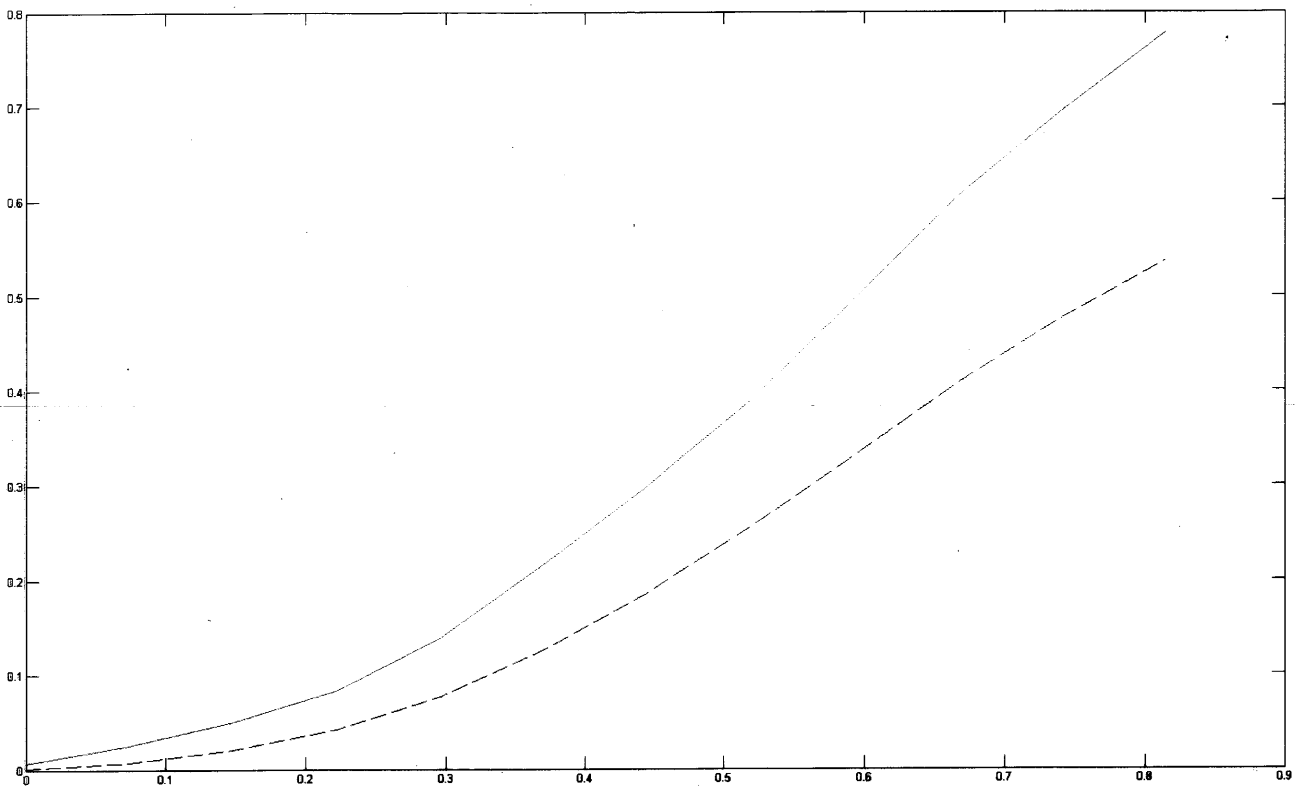
Askel: $\left(\mathbf{M} + \Delta t^2 \beta \mathbf{K}\right)\ddot{\mathbf{q}}_1 = \mathbf{F}_1 - \mathbf{K}\tilde{\mathbf{q}}_1 \Rightarrow \ddot{\mathbf{q}}_1 = \dfrac{F_0}{\rho AL^3}\begin{pmatrix} 1.1640L^2 \\ 4.6991L \end{pmatrix}$

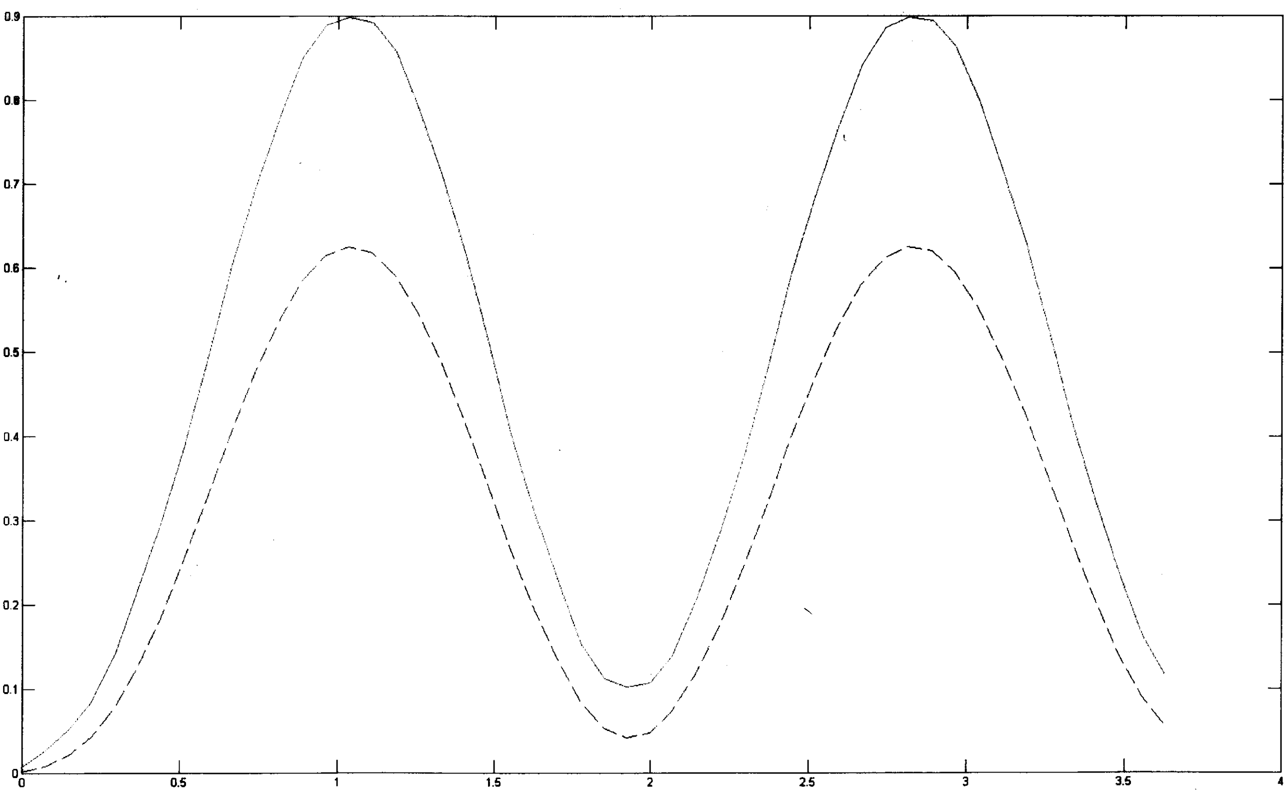
Korjaus:

$$\mathbf{q}_1 = \tilde{\mathbf{q}}_1 + \Delta t^2 \beta \ddot{\mathbf{q}}_1 = \frac{F_0 L^3}{EI}\begin{pmatrix} 0.00159814 \\ 0.00645174/L \end{pmatrix}$$

$$\dot{\mathbf{q}}_1 = \tilde{\dot{\mathbf{q}}}_1 + \Delta t \gamma \ddot{\mathbf{q}}_1 = \frac{F_0}{\sqrt{EI\rho AL^4}}\begin{pmatrix} 0.043130445L^3 \\ 0.174118605L^2 \end{pmatrix}$$

```
beta=1/4;
gamma=1/2;

% ominaiskulmataaajuus
w1=3.5327
%jakson aika
T1=2*pi/w1;
t1=T1/4;
dt=t1/6

% matriisit
K=[12,-6;-6,4];, M=1/420*[156,-22;-22,4];
invi = inv(M+dt^2*beta*K);

% alkuehdot
q0=[0;0];
qdot0=[0;0];

%voima nollassa
F0=[0;0];
q2dot0=M\(F0-K*q0)

% ensimmäinen aika-askel
F(:,1) = [1/6;0];
% ennustajat
q_(:,1)    = q0+dt*qdot0+dt^2/2*(1-2*beta)*q2dot0;
qdot_(:,1)= qdot0+(1-gamma)*dt*q2dot0;
% askel
q2dot(:,1)=invi*(F(:,1)-K*q_(:,1));
q(:,1)= q_(:,1)+beta*dt^2*q2dot(:,1);
qdot(:,1)=qdot_(:,1)+gamma*dt*q2dot(:,1);

for i = 2:12,
    % ensimmäinen aika-askel
    if i < 7,
        F(:,i) = [i/6;0];
    else
        F(:,i) = [1;0];
    end
    % ennustajat
    q_(:,i)    = q(:,i-1)+dt*qdot(:,i-1)+dt^2/2*(1-2*beta)*q2dot(:,i-1);
    qdot_(:,i)= qdot(:,i-1)+(1-gamma)*dt*q2dot(:,i-1);
    % askel
    q2dot(:,i) = invi*(F(:,i)-K*q_(:,i));
    q(:,i)     = q_(:,i)+beta*dt^2*q2dot(:,i);
    qdot(:,i)  = qdot_(:,i)+gamma*dt*q2dot(:,i);
end
```


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4