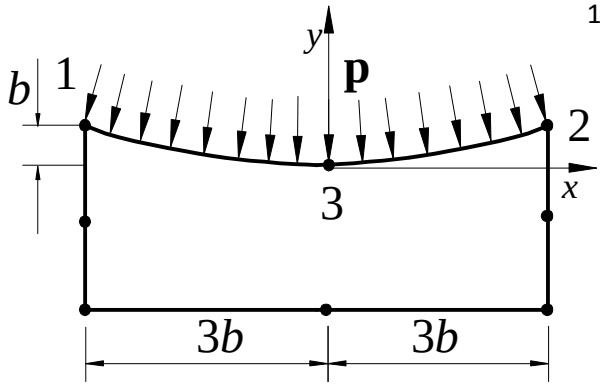



1. Kuvan kvadraattisen elementin sivulle kohdistuu hydrostaattinen painekuorma. Määritä painekuormasta aiheutuva ekvivalenttinen solmukuormitusvektori numeerisesti käyttäen kolmea integrointipistettä. Elementin paksuus on 0.01 m, mitta b on 0.1 m ja paine 10 MPa. Riittää kun lasket vain solmuille 2 ja 3 tulevat komponentit.

Interpolointi ulotetaan reunaviivalle 1-3-2 yksiulotteisena

$$\mathbf{u}(\xi)=\begin{bmatrix}u_s\\ v_s\end{bmatrix}=\begin{bmatrix}N_1^s & 0 & N_2^s & 0 & N_3^s & 0\\ 0 & N_1^s & 0 & N_2^s & 0 & N_3^s\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\\ q_5\\ q_6\end{bmatrix}=\mathbf{N}_s\mathbf{q}_s$$

$$\mathbf{f}^p=\iint_A \mathbf{N}^{\mathrm{T}}\mathbf{p}\,dA$$

$$\mathbf{f}^V=\iiint_V \mathbf{N}^{\mathrm{T}}\mathbf{f}\,dV$$

Reunaviivan pisteet interpoloidaan

$$x_s=N_1^s x_2+N_2^s x_2+N_3^s x_3=\frac{3b}{2}\left[\xi-\xi^2+\xi+\xi^2\right]=3b\xi \qquad \Rightarrow dx=3b\,d\xi=J_x d\xi$$

$$y_s=N_1^s y_1+N_2^s y_2+N_3^s y_3=\frac{b}{2}\left[-\xi+\xi^2+\xi+\xi^2\right]=b\xi^2 \qquad \Rightarrow dy=2b\xi\,d\xi=J_y d\xi$$

$$\Rightarrow ds=\sqrt{J_x^2+J_y^2}\,d\xi=b\sqrt{4\xi^2+9}\,d\xi \qquad dA=t\,ds$$

painepinnan suuntavektori **t** ja normaalivektori **n**

$$\mathbf{t}=\frac{1}{\sqrt{J_x^2+J_y^2}}\begin{bmatrix}J_x\\ J_y\end{bmatrix}=\frac{1}{\sqrt{4\xi^2+9}}\begin{bmatrix}3\\ 2\xi\end{bmatrix}$$

$$\mathbf{n}=\frac{1}{\sqrt{4\xi^2+9}}\begin{bmatrix}-2\xi\\ 3\end{bmatrix} \qquad \Rightarrow \mathbf{p}=-p\,\mathbf{n}=\frac{p}{\sqrt{4\xi^2+9}}\begin{bmatrix}2\xi\\ -3\end{bmatrix}$$

$$\mathbf{f}^p = \iint_A \mathbf{N}^{\mathrm{T}}\mathbf{p}\, dA = t\cdot b\cdot p\int_{-1}^{1}\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}\begin{bmatrix} 2\xi \\ -3 \end{bmatrix}\frac{1}{\sqrt{4\xi^2+9}}\sqrt{4\xi^2+9}\, d\xi = t\cdot b\cdot p\int_{-1}^{1}\begin{bmatrix} 2\xi N_1 \\ -3N_1 \\ 2\xi N_2 \\ -3N_2 \\ 2\xi N_3 \\ -3N_3 \end{bmatrix} d\xi$$

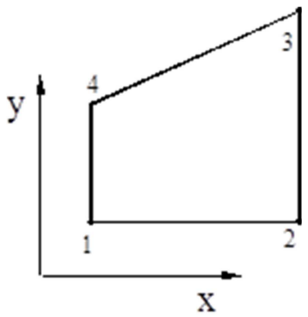
Lasketaan neljä viimeistä komponenttia numeerisesti (muotofunktion N1 arvoja ei siis tarvita)

| ξ | Jx | Jy | N1 | N2 | N3 | Wi | f3 | f4 | f5 | f6 |
|---|---|---|---|---|---|---|---|---|---|---|
| -0.7746 | 0.3 | -0.15492 | 0.687298 | -0.0873 | 0.4 | 0.555556 | 751.3444 | 1454.972 | -3442.65 | -6666.67 |
| 0.0000 | 0.3 | 0 | 0 | 0 | 1 | 0.888889 | 0 | 0 | 0 | -26666.7 |
| 0.7746 | 0.3 | 0.154919 | -0.0873 | 0.687298 | 0.4 | 0.555556 | 5915.322 | -11455 | 3442.652 | -6666.67 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Yht** | | | | | | | **6666.667** | **-10000** | **0** | **-40000** |

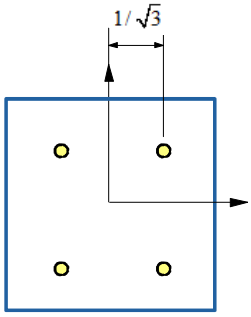
Symmetrian nojalla painesivun ekvivalenttinen (tarkka) kuormitusvektori on

$$\mathbf{f}^p = \begin{bmatrix} -6666.67 \\ -10000 \\ 6666.67 \\ -10000 \\ 0 \\ -40000 \end{bmatrix} N$$




2. Laske kuvan nelisolmuisen elementin polaarinen neliömomentti $I_p = I_x + I_y = \iint_A y^2 dA + \iint_A x^2 dA$ numeerisesti Gauss'in integroinnilla käyttäen 2·2 näytteenottoa.

| Solmu | x [mm] | y [mm] |
|---|---|---|
| 1 | 100 | 100 |
| 2 | 500 | 100 |
| 3 | 500 | 550 |
| 4 | 100 | 400 |




2x2

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} (1-\eta)x_{21} + (1+\eta)x_{34} & (1-\eta)y_{21} + (1+\eta)y_{34} \\ (1-\xi)x_{41} + (1+\xi)x_{32} & (1-\xi)y_{41} + (1+\xi)y_{32} \end{bmatrix}$$

$x_{21} = 400 \qquad x_{34} = 400 \qquad y_{21} = 0 \qquad y_{34} = 150$

$x_{41} = 0 \qquad x_{32} = 0 \qquad y_{41} = 300 \qquad y_{32} = 450 \qquad \Rightarrow$

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} 800 & 150(1+\eta) \\ 0 & 300(1-\xi)+450(1+\xi) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 800 & 150(1+\eta) \\ 0 & 750+150\xi \end{bmatrix} = \begin{bmatrix} 200 & 75(1+\eta)/2 \\ 0 & 75(5+\xi)/2 \end{bmatrix} \Rightarrow J = \det(\mathbf{J}) = 37500 + 7500\xi$$

$$\mathbf{I} \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathbf{f}\left[\mathbf{x}(\xi_i,\eta_j)\right] \det \mathbf{J}(\xi_i,\eta_j)$$

$$y(\xi,\eta) = 100(N_1+N_2) + 400(N_3+N_4) + 150N_3 = 50(1-\eta) + 200(1+\eta) + 150N_3$$
$$= 250 + 150\eta + \frac{150}{4}(1+\xi)(1+\eta)$$

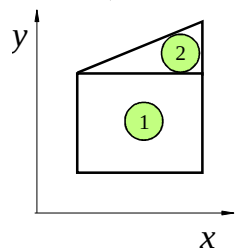
Helposti nähdään, että $x(\xi) = 300 + 200\xi$, joten muodostetaan taulukko integrointipisteistä

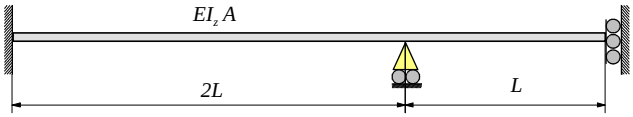
| Piste | ξ | η | J | x | $x^2$ | y | $y^2$ | Ix | Iy |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -0.57735 | -0.57735 | 33169.87 | 184.5299 | 34051.3 | 170.0962 | 28932.7137 | 959694437.1 | 1.13E+09 |
| 2 | 0.57735 | -0.57735 | 41830.13 | 415.4701 | 172615.4 | 188.3975 | 35493.6028 | 1484701913 | 7.22E+09 |
| 3 | 0.57735 | 0.57735 | 41830.13 | 415.4701 | 172615.4 | 429.9038 | 184817.286 | 7730930563 | 7.22E+09 |
| 4 | -0.57735 | 0.57735 | 33169.87 | 184.5299 | 34051.3 | 361.6025 | 130756.397 | 4337173087 | 1.13E+09 |
| | | | | | | | Yht | **14512500000** | **1.67E+10** |

Ip = Ix+Iy = 3.12E+10 $mm^4$

Tarkastus 3.12E+10 $mm^4$

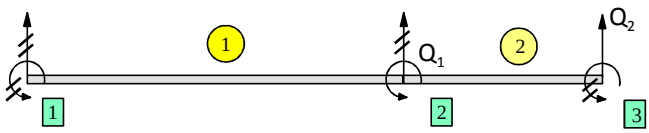
| Elem | y0 | A | Ix0 | Ist | Ix | x0 | Iy0 | Ist | Iy |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 250 | 120000 | 9.00E+08 | 7.5E+09 | 8.4E+09 | 300 | 1.60E+09 | 1.08E+10 | 1.24E+10 |
| 2 | 450 | 30000 | 3.75E+07 | 6.08E+09 | 6.11E+09 | 366.6667 | 2.67E+08 | 4.03E+09 | 4.30E+09 |
| | | | | Yht | **1.45E+10** | | | Yht | **1.67E+10** |






3. Määritä kuvan palkkirakenteen ominaistaajuudet ja ominaismuodot (vektorit). Käytä kahden solmu-vapausasteen palkkielementtejä. Hahmottele myös piirtämällä laskemasi ominaismuodot. Käytä konsistenttia massamatriisia (se täydempi).

| Suure | Arvo | Yksikkö |
|---|---|---|
| ρ | 7850 | $kg/m^3$ |
| *E* | 200 | GPa |
| $I_z$ | $10^{-5}$ | $m^4$ |
| *A* | 0.01 | $m^2$ |
| *L* | 2 | m |




Elementti 1 (vasen palkki)

$$\mathbf{k}_1 = \frac{EI}{8L^3}\begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} \\ \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & 16L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 1 \end{array}$$

Elementti 2 (oikenpuoleinen)

$$\mathbf{k}_2 = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 0 \end{array} \\ \begin{bmatrix} x & x & x & x \\ x & 4L^2 & -6L & x \\ x & -6L & 12 & x \\ x & x & x & x \end{bmatrix} \end{array}\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 0 \end{array}$$

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 6L^2 & -6L \\ -6L & 12 \end{bmatrix} = \frac{6EI}{L^3}\begin{bmatrix} L^2 & -L \\ -L & 2 \end{bmatrix}$$

$$\mathbf{m} = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Palkkien massamatriisit

$$\mathbf{m}_1 = \frac{\rho A \cdot 2L}{420}\begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} \\ \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & 16L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \qquad \mathbf{m}_2 = \frac{\rho A L}{420}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 0 \end{array} \\ \begin{bmatrix} x & x & x & x \\ x & 4L^2 & 13L & x \\ x & 13L & 156 & x \\ x & x & x & x \end{bmatrix} \end{array}\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 0 \end{array} \qquad \Rightarrow$$

$$\mathbf{M} = \frac{\rho A L}{420}\begin{bmatrix} 36L^2 & 13L \\ 13L & 156 \end{bmatrix}$$

Ominaistaajuuksien määritys

$$\det(\mathbf{K}-\lambda\mathbf{M})=0 \Rightarrow \det\left(\begin{bmatrix} L^2 & -L \\ -L & 2 \end{bmatrix} - \lambda \frac{\rho A L^4}{6\cdot 420 EI_z}\begin{bmatrix} 36L^2 & 13L \\ 13L & 156 \end{bmatrix}\right) \Rightarrow \det\left(\begin{bmatrix} L^2(1-36\lambda_0) & -L(1-13\lambda_0) \\ -L(1-13\lambda_0) & 2-156\lambda_0 \end{bmatrix}\right)=0 \Rightarrow$$

$$L^2\left(5447\lambda_0^2 - 254\lambda_0 + 1\right)=0$$

$$\begin{cases} \lambda_{01} = 0.00434115 \\ \lambda_{02} = 0.04229 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0.00434115 \cdot \dfrac{6\cdot 420 EI_z}{\rho A L^4} = 17419.9 \Rightarrow \omega_1 = 131.984 \Rightarrow f_1 = 21.01Hz \\ \lambda_2 = 0.04229 \cdot \dfrac{6\cdot 420 EI_z}{\rho A L^4} = 169699 \Rightarrow \omega_2 = 411.945 \Rightarrow f_2 = 65.56Hz \end{cases}$$

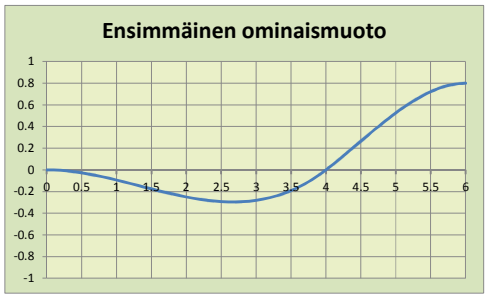
Ominaismuotojen laskenta

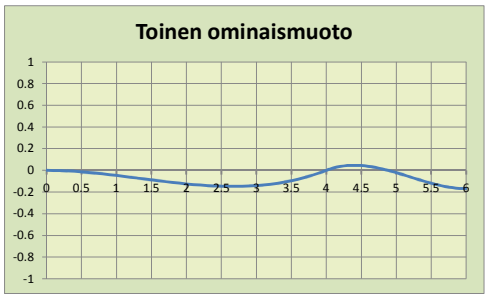
$$\begin{bmatrix} L^2(1-36\lambda_{0i}) & -L(1+13\lambda_{0i}) \\ -L(1+13\lambda_{0i}) & 2-156\lambda_{0i} \end{bmatrix}\begin{bmatrix} 1 \\ x_2^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$(2-156\lambda_{0i})x_2^i = L(1+13\lambda_{0i}) \Rightarrow x_2^i = \frac{L(1+13\lambda_{0i})}{2-156\lambda_{0i}} \Rightarrow$$

$$x_2^1 = \frac{L(1+13\cdot 0.00434115)}{2-156\cdot 0.00434115} = 1.59729 \qquad \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1.59729 \end{bmatrix}$$

$$x_2^2 = \frac{L(1+13\cdot 0.04229)}{2-156\cdot 0.04229} = -0.67422 \qquad \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -0.67422 \end{bmatrix}$$

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*****  INDEX OF DATA SETS ON RESULTS FILE  *****

   SET   TIME/FREQ    LOAD STEP   SUBSTEP  CUMULATIVE
     1   20.999           1          1          1
     2   65.517           1          2          2
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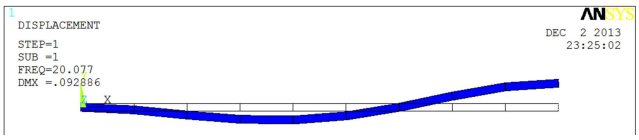


PS. Yllä laskettu toinen ominaistaajuus ja muoto eivät ole järkeviä. Niiden laskeminen edellyttää useamman, kuin kahden elementin laskentamallia. Alla 15 elementtiä, jolloin toinen taajuus on 35.5 Hz.