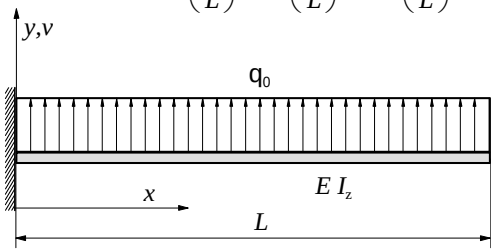
Tampereen teknillinen yliopisto
Teknisen suunnittelun laitos

EDE-21100 Elementtimenetelmän perusteet. Harjoitus 4 Syksy 2013.

1. Laske kuvan palkin (E Iz) siirtymäfunktion v(x) likiarvo potentiaalienergian minimin periaatteella ja vertaa pään siirtymää tarkkaan arvoon (harj. 3) käyttämällä kinemaattisesti käypää yritettä. Valitse yrite polynomeista:

$$p(x) = \alpha_0 + \alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3$$



$$v_m(\alpha_2,\alpha_3,x,L) := \alpha_2\cdot\left(\frac{x}{L}\right)^2 + \alpha_3\cdot\left(\frac{x}{L}\right)^3$$

$$v_{xx}(\alpha_2,\alpha_3,x,L) := \frac{d^2}{dx^2}v_m(\alpha_2,\alpha_3,x,L) \rightarrow \frac{2\cdot\alpha_2}{L^2} + \frac{6\cdot\alpha_3\cdot x}{L^3}$$

$$U_p(E,I_z,\alpha_2,\alpha_3,L) := \frac{E\cdot I_z}{2}\cdot\int_0^L \left(v_{xx}(\alpha_2,\alpha_3,x,L)\right)^2 dx \rightarrow \frac{2\cdot E\cdot I_z\cdot\left(\alpha_2^2 + 3\cdot\alpha_2\cdot\alpha_3 + 3\cdot\alpha_3^2\right)}{L^3}$$

$$WP(q_0,\alpha_2,\alpha_3,L) := -q_0\cdot\int_0^L v_m(\alpha_2,\alpha_3,x,L)\,dx \rightarrow -\frac{L\cdot q_0\cdot(4\cdot\alpha_2 + 3\cdot\alpha_3)}{12}$$

$$\Pi(q_0,E,I_z,\alpha_2,\alpha_3,L) := U_p(E,I_z,\alpha_2,\alpha_3,L) + WP(q_0,\alpha_2,\alpha_3,L)$$

$$\Pi(q_0,E,I_z,\alpha_2,\alpha_3,L) \rightarrow \frac{2\cdot E\cdot I_z\cdot\left(\alpha_2^2 + 3\cdot\alpha_2\cdot\alpha_3 + 3\cdot\alpha_3^2\right)}{L^3} - \frac{L\cdot q_0\cdot(4\cdot\alpha_2 + 3\cdot\alpha_3)}{12}$$

given

$$\frac{d}{d\alpha_2}\Pi(q_0,E,I_z,\alpha_2,\alpha_3,L) = 0$$

$$\frac{d}{d\alpha_3}\Pi(q_0,E,I_z,\alpha_2,\alpha_3,L) = 0$$

$$\alpha(q_0,E,I_z,\alpha_2,\alpha_3,L) := \mathrm{find}(\alpha_2,\alpha_3) \rightarrow \begin{pmatrix} \frac{5\cdot L^4\cdot q_0}{24\cdot E\cdot I_z} \\ -\frac{L^4\cdot q_0}{12\cdot E\cdot I_z} \end{pmatrix}$$

$$a_3(q_0,E,I_z,L) := -\frac{L^4\cdot q_0}{12\cdot E\cdot I_z}$$

$$a_2(q_0,E,I_z,L) := \frac{5\cdot L^4\cdot q_0}{24\cdot E\cdot I_z}$$

$$a_2(q_0,E,I_z,L) + a_3(q_0,E,I_z,L) \rightarrow \frac{L^4\cdot q_0}{8\cdot E\cdot I_z}$$

Taas tuli tarkka pään siirtymä

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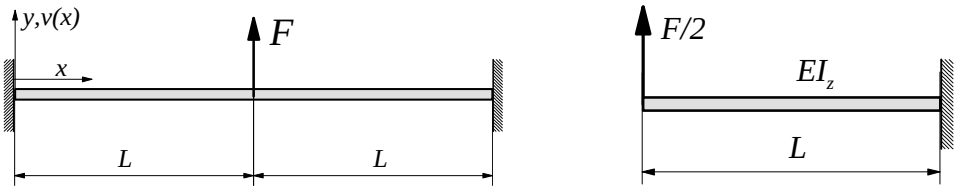
2. Laske kuvan palkkien ($E\ I_z$) siirtymäfunktion $v(x)$ likiarvo potentiaalienergian minimin periaatteella käyttämällä kinemaattisesti käypää yritettä rakenteen puolikkaalle. Valitse likiarvo polynomeista $p(x) = \alpha_0 + \alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3$



symmetria $\Rightarrow \alpha_1 = 0 \quad \tilde{v}(x) = \alpha_0 + \alpha_2\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3$

$$\tilde{v}(L) = \alpha_0 + \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_0 = -\alpha_2 - \alpha_3$$

$$\tilde{v}_{,x} = \alpha_2 \frac{2x}{L^2} + \alpha_3 \frac{3x^2}{L^3} \Rightarrow \tilde{v}_{,x}(L) = 2\frac{\alpha_2}{L} + 3\frac{\alpha_3}{L} = 0 \Rightarrow \alpha_2 = -\frac{3}{2}\alpha_3$$

$$\alpha_0 = -\alpha_2 - \alpha_3 = \frac{3}{2}\alpha_3 - \alpha_3 = \frac{\alpha_3}{2}$$

$$\tilde{v}(x) = \frac{\alpha_3}{2} - \frac{3}{2}\alpha_3\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3 = \frac{\alpha_3}{2}\left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right]$$

$$\tilde{v}_{,x}(x) = \frac{\alpha_3}{2}\left[-6\frac{x}{L^2} + 6\frac{x^2}{L^3}\right] \qquad \tilde{v}_{,xx}(x) = \frac{\alpha_3}{L^3}\left[6x - 3L\right] \qquad \tilde{v}_{,xx}^2 = \frac{\alpha_3^2}{L^6}\left[36x^2 - 36Lx + 9L^2\right]$$

$$U = \frac{EI_z}{2L^6}\alpha_3^2 \int_0^L \left[36x^2 - 36Lx + 9L^2\right] dx = \frac{EI_z}{2L^6}\alpha_3^2 \left[12x^3 - 18Lx^2 + 9L^2 x\right]_0^L = \frac{3EI_z}{2L^3}\alpha_3^2$$

$$WP = -\frac{F}{2}\alpha_0 = -\frac{F}{4}\alpha_3$$

$$\Pi = U + WP \rightarrow \min \Rightarrow \frac{\partial U}{\partial \alpha_3} = -\frac{\partial WP}{\partial \alpha_3} \Rightarrow \frac{3EI_z}{L^3}\alpha_3 = \frac{F}{4} \Rightarrow \alpha_3 = \frac{F\,L^3}{12EI_z}$$

Taipuma keskellä

$$\tilde{v}(0) = \alpha_0 = \frac{F\,L^3}{24EI_z}$$

joka on tarkka ratkaisu (Lujuusopin perusteet)

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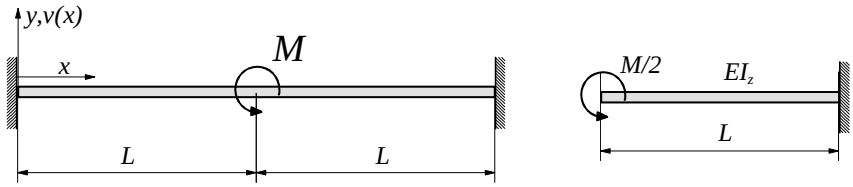

ei pystykuormia $\Rightarrow \alpha_0 = 0 \quad \tilde{v}(x) = \alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2 + \alpha_3\left(\frac{x}{L}\right)^3$

$$\tilde{v}(L) = \alpha_1 + \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_2 - \alpha_3$$

$$\tilde{v}_{,x}(x) = \frac{\alpha_1}{L} + \frac{2\alpha_2 x}{L^2} + \frac{3\alpha_3 x^2}{L^3} \Rightarrow \tilde{v}_{,x}(L) = \frac{\alpha_1}{L} + \frac{2\alpha_2}{L} + \frac{3\alpha_3}{L} = 0 \Rightarrow -\alpha_2 - \alpha_3 + 2\alpha_2 + 3\alpha_3 = 0$$

$$\Rightarrow \alpha_2 = -2\alpha_3 \qquad \alpha_1 = \alpha_3$$

$$\tilde{v}(x) = \alpha_3\left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right] \qquad \tilde{v}_{,x}(x) = \alpha_3\left[\frac{1}{L} - 4\frac{x}{L^2} + 3\frac{x^2}{L^3}\right]$$

$$\tilde{v}_{,xx}(x) = \alpha_3\left[-\frac{4}{L^2} + \frac{6x}{L^3}\right] \qquad \tilde{v}_{,xx}^2 = \frac{\alpha_3^2}{L^6}\left[36x^2 - 48Lx + 16L^2\right]$$

$$U = \frac{EI_z}{2L^6}\alpha_3^2\int_0^L\left[36x^2 - 48Lx + 16L^2\right]dx = \frac{EI_z}{2L^6}\alpha_3^2\left[12x^3 - 24Lx^2 + 16L^2x\right]_0^L = \frac{2EI_z}{L^3}\alpha_3^2$$

$$WP = -\frac{M}{2}\tilde{v}_{,x}(0) = -\frac{M}{2L}\alpha_3$$

$$\Pi = U + WP \rightarrow \min \Rightarrow \frac{\partial U}{\partial \alpha_3} = -\frac{\partial WP}{\partial \alpha_3} \Rightarrow \frac{4EI_z}{L^3}\alpha_3 = \frac{M}{2L} \Rightarrow \alpha_3 = \frac{M\,L^2}{8EI_z}$$

Tark. Haetaan taipuman maksimi

$$\tilde{v}_{,x}(x) = \alpha_3\left[\frac{1}{L} - 4\frac{x}{L^2} + 3\frac{x^2}{L^3}\right] = 0 \Rightarrow L^2 - 4Lx + 3x^2 = 0$$

$$x_{1,2} = \frac{+4L \pm \sqrt{16L^2 - 4\cdot 3\cdot L^2}}{6} \Rightarrow \begin{cases} x = L \\ x = L/3 \end{cases}$$

$$\tilde{v}(L/3) = \frac{M\,L^2}{8EI_z}\left[\frac{1}{3} - \frac{2}{9} + \frac{1}{27}\right] = \frac{M\,L^2}{54EI_z}$$

joka on tarkka ratkaisu (Lujuusopin perusteet)

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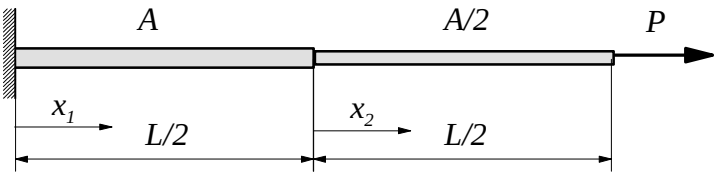
3. Kuvan molempien sauvojen kimmokerroin on E. Laske kuvan sauvarakenteen pään siirtymä $u(L)$ ja molempien sauvojen jännitykset potentiaalienergian $\Pi = U + WP$ minimin periaatteella käyttämällä paksumalle osalle kinemaattisesti käypää ($\tilde{u}_1(0) = 0$) siirtymäestimaattia $\tilde{u}_1(x_1) = Q_1\left(\frac{2x_1}{L}\right)$ ja ohuemmalle osalle estimaattia $\tilde{u}_2(x_2) = Q_1\left(1 - \frac{2x_2}{L}\right) + Q_2\left(\frac{2x_2}{L}\right)$. Vertaa tuloksia tarkkoihin arvoihin.



$$\tilde{u}_{1,x} = Q_1\left(\frac{2}{L}\right) \Rightarrow \tilde{u}_{1,x}^2 = \frac{4Q_1^2}{L^2} \qquad U_1 = \frac{EA}{2}\int_0^{L/2} \frac{4Q_1^2}{L^2}dx = \frac{EA}{L}Q_1^2$$

$$\tilde{u}_{2,x} = \frac{2Q_2 - 2Q_1}{L} \Rightarrow \tilde{u}_{2,x}^2 = \frac{4}{L^2}\left[Q_1^2 - 2Q_1Q_2 + Q_2^2\right] U_2 = \frac{EA}{L^2}\left[Q_1^2 - 2Q_1Q_2 + Q_2^2\right]\int_0^{L/2} dx = \frac{EA}{2L}\left[Q_1^2 - 2Q_1Q_2 + Q_2^2\right]$$

$$U = U_1 + U_2 = \frac{EA}{2L}\left[3Q_1^2 - 2Q_1Q_2 + Q_2^2\right] \qquad WP = -PQ_2 \qquad \Pi = U + WP \to \min \Rightarrow$$

$$\frac{\partial U}{\partial Q_1} = 0 = \frac{EA}{2L}\left[6Q_1 - 2Q_2\right]$$

$$\frac{\partial U}{\partial Q_2} = \frac{EA}{2L}\left[-2Q_1 + 2Q_2\right] = -\frac{\partial WP}{\partial Q_2} = P$$

$$\frac{EA}{L}\left[3Q_1 - Q_2\right] = 0$$

$$\frac{EA}{L}\left[-Q_1 + Q_2\right] = P$$

$$\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{PL}{EA} \quad \Rightarrow \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{PL}{EA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{PL}{2EA}$$

$$\tilde{u}_{1,x} = Q_1\left(\frac{2}{L}\right) \Rightarrow \sigma_1 = E\,\tilde{u}_{1,x} = E\frac{P\,L\,2}{2E\,A\,L} = \frac{P}{A}$$

$$\tilde{u}_{2,x} = \frac{2Q_2 - 2Q_1}{L} \Rightarrow \sigma_2 = E\,\tilde{u}_{2,x} = E\frac{2}{L}\frac{PL}{EA} = \frac{2P}{A}$$

siirtymät Q_1 ja Q_2 ovat myös täysin tarkat