Introduction to materials modelling

5. exercise – isotropic elastic material model

1. The strain energy W of a linear isotropic elastic solid can be written in the following equivalent forms:

$$W_1 = \tfrac{1}{2}a_1 I_1^2 + b_1 I_2,$$
$$W_2 = \tfrac{1}{2}a_2 I_1^2 + b_2 J_2,$$
$$W_3 = \tfrac{1}{2}a_3 I_1^2 + b_3 \tilde{I}_2,$$

   where I_1, I_2 are the principal invariants of the infinitesimal strain tensor, $I_1 = \operatorname{tr}\boldsymbol{\varepsilon} = \varepsilon_{kk}, I_2 = \frac{1}{2}(\operatorname{tr}(\boldsymbol{\varepsilon}^2) - I_1^2)$ and J_2 is the second invariant of the deviatoric strain tensor $\mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3}I_1\mathbf{I}$, i.e. $J_2 = \frac{1}{2}\operatorname{tr}(\mathbf{e}^2)$ and $\tilde{I}_2$ is a generic quadratic invariant $\tilde{I}_2 = \frac{1}{2}\operatorname{tr}(\boldsymbol{\varepsilon}^2)$.
   Determine the coefficients a_i, b_i in terms of Lamé parameters $\lambda, \mu = G$ and also in terms of the Young's, shear and bulk modulae E, G and K, respectively.

2. The most general isotropic elastic material model is of the form

$$\boldsymbol{\sigma} = a_0\mathbf{I} + a_1\boldsymbol{\varepsilon} + a_2\boldsymbol{\varepsilon}^2,$$

   where the coefficients a_i can depend on the invariants of the strain tensor $\boldsymbol{\varepsilon}$. The strain energy w with respect to unit volume can also be written as a function of the invariants of the strain tensor and its deviator I_1, J_2 and $J_3 = \det\mathbf{e}$ as

$$W = W(I_1, J_2, J_3).$$

   (a) Determine the coefficients a_i expressed in terms of the derivatives of W.

   (b) If the material model is expressable in the form $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ the coefficient $a_2 \equiv 0$. Formulate a non-linear isotropic constitutive model in terms of bulk modulus K and shear modulus G, which are functions of the invariants I_1 and J_2

$$K = K(I_1, J_2) \qquad \text{and} \qquad G = G(I_1, J_2).$$

   For hyperelasticity, the functions $K(I_1, J_2)$ and $G(I_1, J_2)$ cannot be independent. Derive the conditions which they have to obey.

   (c) A good approximation for certain metallic materials is to assume that the volumetrix behaviour is linear and the shear modulus depends only on the second invariant of the deviatoric strain. Thus the volumetric and deviatoric behaviour is uncoupled. Assume the following relation to the shear modulus

$$G(J_2) = G_0(1 + \alpha J_2),$$

   where α is a dimensonless material parameter. Determine the strain energy function W. Find out the stress-strain relation and draw it in a unixaxial tension for different values of α.

Hint: If the second order tensor $\mathbf{A}$ is deviatoric, i.e. $\operatorname{tr}\mathbf{A} = 0$, then $J_3 = \det\mathbf{A} = \frac{1}{3}\operatorname{tr}(\mathbf{A}^3)$.