FEM advanced course

Lecture 11 - Integration of material models

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Plasticity - basic incredients

Characteristic feature in plastic deformation is the formation of permanent deformations.

In a closed loading process energy is dissipated into structural changes of a material and into heat.

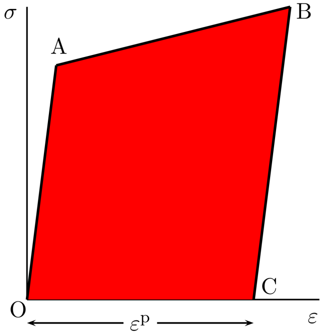
To discribe plasticity three type of equations are needed:

1. **yield condition** which determines the boundary of the elastic domain,
2. **flow rule** which describes how the plastic strains evolve,
3. **hardening rule** which describes the evolution of the elastic domain, i.e. the evolution of the yield surface.

Permanent plastic strains

Stress loading OABC:
Elastic deformation OA
Plastic deformation AB
Elastic deformaton during unloading BC

Dissipated energy $\int \sigma \mathrm{d}\varepsilon^{\mathrm{p}}$ in the cycle OABC.



Relation between tangent- and hardening modulae

Small strains can be addidively decomposed into elastic and plastic components

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}.$$

Tangent and hardening modulae are defined as

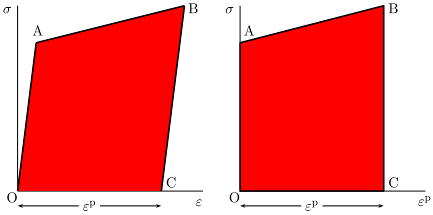
$$E_{\mathrm{t}} = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}, \quad E_{\mathrm{p}} = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon^{\mathrm{p}}}.$$

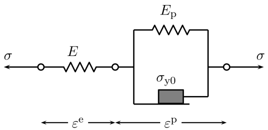
For the increments

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon^{\mathrm{e}} + \mathrm{d}\varepsilon^{\mathrm{p}} \quad \Rightarrow \quad \frac{\mathrm{d}\sigma}{E_{\mathrm{t}}} = \frac{\mathrm{d}\sigma}{E} + \frac{\mathrm{d}\sigma}{E_{\mathrm{p}}},$$

yielding

$$E_{\mathrm{t}} = \frac{EE_{\mathrm{p}}}{E + E_{\mathrm{p}}} \quad \text{or} \quad E_{\mathrm{p}} = \frac{EE_{\mathrm{t}}}{E - E_{\mathrm{t}}}.$$





Physics of plastic deformation

Deformation of polycrystals:

- First slip in crystals with slip planes oriented at 45° angle to the direction of applied stress.
- Initial yield stress depends on the grain size: Hall-Petch relation

$$\sigma_{y0} = \sigma_0 + \frac{k}{\sqrt{d}}.$$

- Increased dislocation density causes increase in the slip deformation resistance which shows in hardening response.

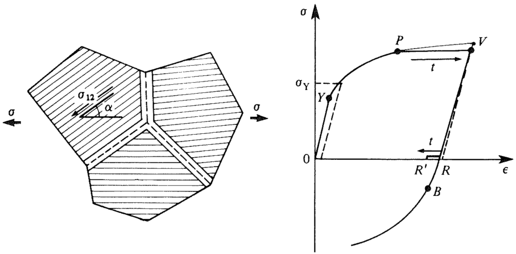


Figure 1.17 from Lemaitre & Chaboche: *Mechanics of Solid Materials*

Yield function (initial)

Written usually as $f(\boldsymbol{\sigma}, \text{parameters}) = 0$.

Separates the elastic domain from the plastic state:

$$
\begin{aligned}
f(\boldsymbol{\sigma}, ..) < 0 \quad & \text{stresss in the elastic domain} \\
f(\boldsymbol{\sigma}, ..) = 0 \quad & \text{plastic state} \\
f(\boldsymbol{\sigma}, ..) > 0 \quad & \text{not possible}
\end{aligned}
$$

For an isotropic solid the yield function has to be independent of coordinate orientation, i.e.

$$
f(\boldsymbol{\sigma}, ..) = f(\boldsymbol{\beta}\boldsymbol{\sigma}\boldsymbol{\beta}^{\mathrm{T}}, ..) \quad \forall \, \text{orthogonal} \, \boldsymbol{\beta}
$$

Thus $f(I_1, I_2, I_3)$ or preferably $f(I_1, J_2, \cos 3\theta)$

Yield function (cont'd)

If an isotropic yield function is given in the form

$$f(I_1, J_2, \cos 3\theta) = 0,$$

it facilitates the investigation of its symmetry properties in the deviatoric plane.

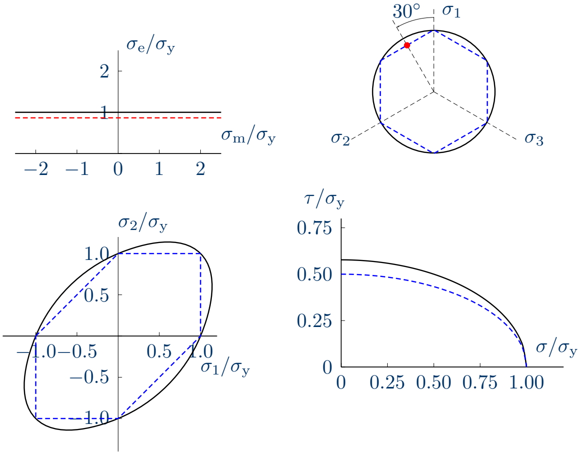
- The yield function is 120° periodic, i.e. $\rho = \sqrt{2J_2}$ has to have same values at θ and $\theta + 120^\circ$.
- Since $\cos$ is an even function, there has to be symmetry with respect to $\theta = 0^\circ$. Due to the 120° periodicity, f has to be symmetric also wrt $\theta = 120^\circ$ and $\theta = 240^\circ$.
- If we set $\theta = 60^\circ + \psi$, then $\cos(3\theta) = -\cos(3\psi)$ and setting $\theta = 60^\circ - \psi$ gives $\cos(3\theta) = -\cos(3\psi)$, so they have the same ρ, thus the yield curve at deviatoric plane is symmetric about $\theta = 60^\circ$, thus it has to be symmetric also about $\theta = 180^\circ$ and $\theta = 300^\circ$.

As a conclusion the initial yield curve for isotropic solids in the deviatoric plane is completely characterized by its form in the sector $0^\circ \leq \theta \leq 60^\circ$.

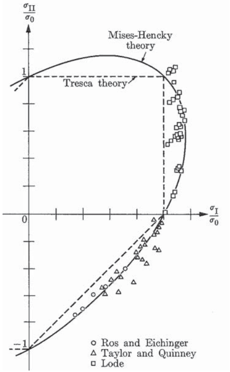
Some well known yield functions

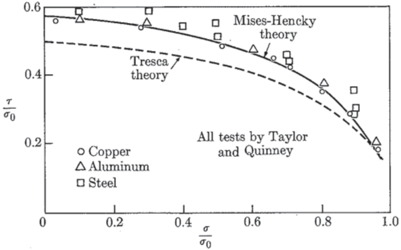
1. Pressure independent yield functions $f(J_2, \cos 3\theta) = 0$:
   - Tresca
     $$\tau_{\text{max}} = \tfrac{1}{2}(\sigma_1 - \sigma_3) - \tau_{\text{y}} = 0$$
   - von Mises
     $$\sqrt{3J_2} - \sigma_{\text{y}} = 0, \qquad \text{or} \qquad \sqrt{J_2} - \tau_{\text{y}} = 0.$$
2. Pressure dependent yield functions $f(I_1, J_2, \cos 3\theta) = 0$:
   - Drucker-Prager
     $$\sqrt{3J_2} + \alpha I_1 - \beta = 0$$
   - Mohr-Coulomb
     $$m\sigma_1 + \sigma_3 - \sigma_{\text{c}} = 0$$

Tresca vs. von Mises yield surfaces

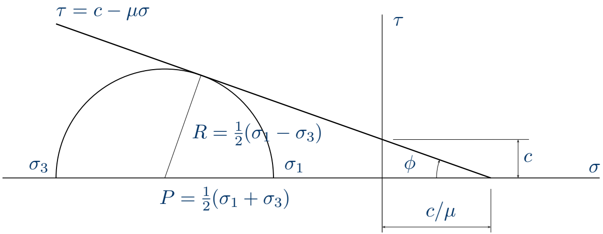


Tresca vs. von Mises - experiments

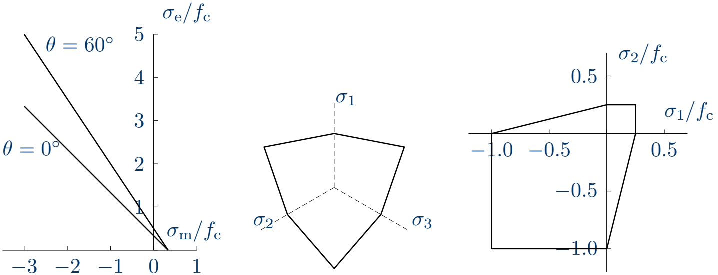




Mohr-Coulomb yield criteria

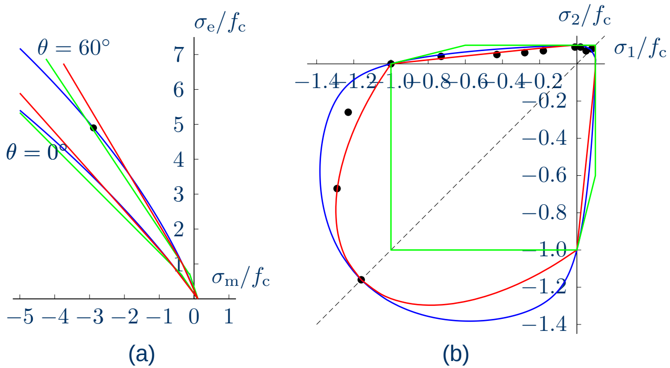


Mohr-Coulomb yield criteria (cont'd)



f_c is the uniaxial compressive strength.

Failure surfaces for concrete



Mohr-Coulomb with tension cut-off (green), Barcelona model (red), Ottosen's model (blue).

Black dots are test results by Kupfer et al. *J. Am. Concr. Inst.*, 66 (1969), pp. 656-666

Solution of elasto-plastic material model

Assume, that at time t_n stresses $\boldsymbol{\sigma}_n$, strains $\boldsymbol{\varepsilon}_n$ and plastic strains $\boldsymbol{\varepsilon}_n^{\mathrm{p}}$ are know. The task is to solve the following equations system at time t_{n+1}

$$\begin{cases} \boldsymbol{\sigma}_{n+1} = \boldsymbol{C}^{\mathrm{e}}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_{n+1}^{\mathrm{p}}) \\ f(\boldsymbol{\sigma}_{n+1}, \lambda_{n+1}) = 0 \\ \dot{\boldsymbol{\varepsilon}}_{n+1}^{\mathrm{p}} = \dot{\lambda}_{n+1} \left.\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}_{n+1}} = \dot{\lambda}_{n+1} \boldsymbol{n}_{n+1} \end{cases}$$

Rate form of the constitutive equation

$$\dot{\boldsymbol{\sigma}}_{n+1} = \boldsymbol{C}^{\mathrm{e}}(\dot{\boldsymbol{\varepsilon}}_{n+1} - \dot{\boldsymbol{\varepsilon}}_{n+1}^{\mathrm{p}}) \quad \Rightarrow \quad \frac{\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n}{\Delta t} = \boldsymbol{C}^{\mathrm{e}} \left(\frac{\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n}{\Delta t} - \frac{\lambda_{n+1} - \lambda_n}{\Delta t} \boldsymbol{n}_{n+1} \right)$$

$$\Rightarrow \quad \Delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{e}}(\Delta\boldsymbol{\varepsilon} - \Delta\lambda \boldsymbol{n})$$

Solution of elasto-plastic material model - linearization

Linearizing wrt the state $\boldsymbol{\sigma}^i_{n+1}, \lambda^i_{n+1}$

$$\delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{e}}(\delta\boldsymbol{\varepsilon} - \delta\lambda\boldsymbol{n} + \Delta\lambda\delta\boldsymbol{n}), \quad \text{now} \quad \delta\boldsymbol{n} = \frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\sigma}}\delta\boldsymbol{\sigma} + \frac{\partial\boldsymbol{n}}{\partial\lambda}\delta\lambda$$

$$\left[(\boldsymbol{C}^{\mathrm{e}})^{-1} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\sigma}}\right]\delta\boldsymbol{\sigma} = \delta\boldsymbol{\varepsilon} - \left(\boldsymbol{n} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\lambda}\right)\delta\lambda \tag{1}$$

The yield and consistency conditions are not necessarily satisfied at state $\boldsymbol{\sigma}^i_{n+1}, \lambda^i_{n+1}$, linearizing the yield function

$$f(\boldsymbol{\sigma}^{i+1}_{n+1}, \lambda^{i+1}_{n+1}) = f(\boldsymbol{\sigma}^i_{n+1}, \lambda^i_{n+1}) + \frac{\partial f}{\partial\boldsymbol{\sigma}}\delta\boldsymbol{\sigma} + \frac{f}{\partial\lambda}\delta\lambda \approx 0 \tag{2}$$

For simplicity, denote $f^i = f(\boldsymbol{\sigma}^i_{n+1}, \lambda^i_{n+1})$. Substituting the change in stress (1) into (2) and denoting $\boldsymbol{D} = (\boldsymbol{C}^{\mathrm{e}})^{-1} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\sigma}}$, it is obtained

$$f^i + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}^{-1}\left[\delta\boldsymbol{\varepsilon} - \left(\boldsymbol{n} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\lambda}\right)\delta\lambda\right] + \frac{\partial f}{\partial\lambda}\delta\lambda = 0$$

$$\Rightarrow \quad \delta\lambda = \frac{1}{A}(f^i + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}^{-1}\delta\boldsymbol{\varepsilon}), \quad \text{where} \quad A = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}^{-1}\left(\boldsymbol{n} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\lambda}\right) - \frac{\partial f}{\partial\lambda}\delta\lambda. \tag{3}$$

Algorithmic tangent

The algorithmic tangent is computed after the iteration is converged, then $f^k = 0$ and substituting (3) into (1) gives

$$\delta\boldsymbol{\sigma} = \left[\boldsymbol{D}^{-1} - \frac{1}{A}\boldsymbol{D}^{-1}\left(\boldsymbol{n} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\lambda}\right)\boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}^{-1}\right]\delta\boldsymbol{\varepsilon} \quad \Rightarrow \quad \delta\boldsymbol{\sigma} = \boldsymbol{C}^{ATS}\delta\boldsymbol{\varepsilon},$$

where

$$\boldsymbol{C}^{ATS} = \boldsymbol{D}^{-1} - \frac{1}{A}\boldsymbol{D}^{-1}\left(\boldsymbol{n} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\lambda}\right)\boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}^{-1}, \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{C}^{\mathrm{e}} + \Delta\lambda\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\sigma}}.$$

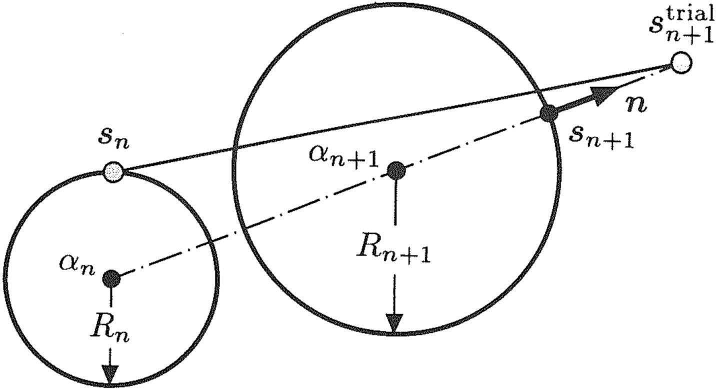
The ATS goes to the stiffness matrix

$$\boldsymbol{K}_0^{(\mathrm{e})} = \int_{\Omega^{(\mathrm{e})}} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{ATS}\boldsymbol{B}\,\mathrm{d}V,$$

and is necessary to obtain quadratic convergence in the solution of equilibrium equations of the system.

Geometric illustration of the radial return algorihm

Both kinematic and isotropic hardening (Fig. Simo, Hughes *Computational Inelasticity*, Springer 2000)



Summary

1. Initial values $\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_n^{\mathrm{p}}, \lambda_n$ and $\boldsymbol{C}^{\mathrm{e}}$, new strain $\boldsymbol{\varepsilon}_{n+1}$.
2. Compute the elastic predictor: $\boldsymbol{\sigma}_{n+1}^{\mathrm{e}} = \boldsymbol{C}^{\mathrm{e}}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^{\mathrm{p}})$.
3. Check if the yield condition is satisfied.
   - (i) If $f(\boldsymbol{\sigma}_{n+1}^{\mathrm{e}}, \lambda_n) < 0$ then the state at t_{n+1} is elastic, set $\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{\mathrm{e}}, \lambda_{n+1} = \lambda_n, \boldsymbol{\varepsilon}_{n+1}^{\mathrm{p}} = \boldsymbol{\varepsilon}_n^{\mathrm{p}}$ and $\boldsymbol{C} = \boldsymbol{C}^{\mathrm{e}}$ and exit.
   - (ii) $f(\boldsymbol{\sigma}_{n+1}^{\mathrm{e}}, \lambda_n) \geq 0$ then the state is plastic, solve $\boldsymbol{\sigma}_{n+1}, \lambda_{n+1}$ iterating:
     - (a) $\delta\lambda = A^{-1} f(\boldsymbol{\sigma}_{n+1}^{i}, \lambda_{n+1}^{i})$,
     - (b) $\delta\boldsymbol{\sigma} = -\boldsymbol{D}^{-1}(\boldsymbol{n} + \Delta\lambda \dfrac{\partial \boldsymbol{n}}{\partial \lambda})\delta\lambda$,
     - (c) update: $\boldsymbol{\sigma}_{n+1}^{i+1} = \boldsymbol{\sigma}_{n+1}^{i} + \delta\boldsymbol{\sigma}$ and $\lambda_{n+1}^{i+1} = \lambda_{n+1}^{i} + \delta\lambda$
   - (iii) If convergence obtained, then compute the algorithmic tangent matrix.