FEM advanced course

Lecture 8 - Plate and shell elements, 3D continuum element

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Plate models - kinematical assumptions

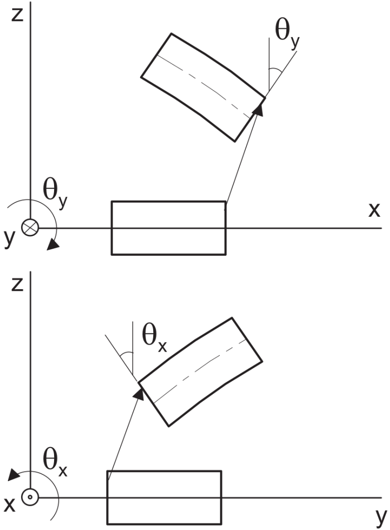
Assume the plate midsurface to be the (x, y)-plane. The kinematical assumptions for thin-plates are:

1. the normals to the midsurface remains straight during deformation,
2. the fibres normal to the midsurface are inextensional.

These assumption can be satisfied with the following displacement field

$$
\begin{aligned}
u(x, y, z) &= z\theta_y(x, y), \\
v(x, y, z) &= -z\theta_x(x, y), \\
w(x, y, z) &= w_c(x, y).
\end{aligned}
$$

They are know as the Reissner-Mindlin plate model kinematical relations.



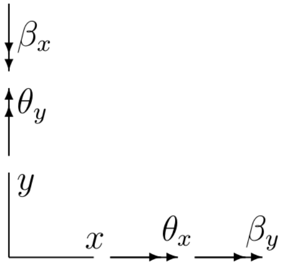
Kinematical asumptions - cont'd

Defining new notation for rotations $\beta_x = -\theta_y, \quad \beta_y = \theta_x$, and the displacement field can be written as

$$\begin{aligned} u(x,y,z) &= -z\beta_x(x,y), \\ v(x,y,z) &= -z\beta_y(x,y), \\ w(x,y,z) &= w(x,y). \end{aligned}$$

The strains are

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -z\beta_{x,x} = z\kappa_x, \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -z\beta_{y,y} = z\kappa_y, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\beta_{x,y} + \beta_{y,x}\right) = z\kappa_{xy}, \\ \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = w_{,x} - \beta_x, \\ \gamma_{zy} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = w_{,y} - \beta_y. \end{aligned}$$



Stress resultants, curvatures

Bending moments and the twisting moment are defined as

$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_x dz, \qquad M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_y dz, \qquad M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\tau_{xy} dz,$$

and the transverse shear forces as

$$Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz, \qquad Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz.$$

Collecting "curvatures", shear deformations, moments and shear forces

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} \end{bmatrix}^T \qquad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_{xz} & \gamma_{yz} \end{bmatrix}^T, \qquad \boldsymbol{m} = \begin{bmatrix} M_x & M_y & M_{xy} \end{bmatrix}^T, \qquad \boldsymbol{q} = \begin{bmatrix} Q_x & Q_y \end{bmatrix}^T.$$

Stress resultants

Plane stress conditions assumed, thus

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{array} \right] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\}, \quad \Rightarrow \quad \begin{array}{rcl} M_x & = & D(\kappa_x + \nu\kappa_y), \\ M_y & = & D(\kappa_y + \nu\kappa_x), \\ M_{xy} & = & \frac{1}{2}D(1-\nu)\kappa_{xy} \end{array} ,$$

where D is the bending rigidity of the plate

$$D = \frac{Et^3}{12(1-\nu^2)}.$$

$$\boldsymbol{m} = \boldsymbol{D}_b \boldsymbol{\kappa}, \quad \boldsymbol{D}_b = D \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{array} \right].$$

Shear forces are

$$\boldsymbol{q} = \boldsymbol{D}_s \boldsymbol{\gamma}, \qquad \boldsymbol{D}_s = kGt \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right],$$

where k is the shear correction factor.

Virtual work of the Reissner-Mindlin model

$$-\int_A \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}\right) dzdA$$

$$+ \int_A f\delta w dA + \int_{S_\sigma} (\bar{Q}_n \delta w - \bar{M}_n \delta\beta_n - \bar{M}_{ns}\delta\beta_s) ds = 0.$$

Integrating the thickness direction gives

$$-\int_A \left(M_x \delta\kappa_x + M_y \delta\kappa_y + M_{xy}\delta\kappa_{xy} + Q_x \delta\gamma_{xz} + Q_y \delta\gamma_{yz}\right) dA$$

$$+ \int_A f\delta w dA + \int_{S_\sigma} (\bar{Q}_n \delta w - \bar{M}_n \delta\beta_n - \bar{M}_{ns}\delta\beta_s) ds = 0.$$

FEM for the Reissner-Mindlin model

C_0-continuity is enough for all functions w, β_x and β_y. For a n-node element, the interpolation is

$$w = \sum_{i=1}^{n} N_i w_i, \qquad \beta_x = \sum_{i=1}^{n} N_i \beta_{xi}, \qquad \beta_y = \sum_{i=1}^{n} N_i \beta_{yi}.$$

The curvature vectors is thus

$$\boldsymbol{\kappa} = \left\{ \begin{array}{c} -\beta_{x,x} \\ -\beta_{y,y} \\ -\beta_{x,y} - \beta_{y,x} \end{array} \right\} = \sum_{i=1}^{n} \left[\begin{array}{ccc} 0 & -N_{i,x} & 0 \\ 0 & 0 & -N_{i,y} \\ 0 & -N_{i,y} & -N_{i,x} \end{array} \right] \left\{ \begin{array}{c} w_i \\ \beta_{xi} \\ \beta_{yi} \end{array} \right\},$$

in short

$$\boldsymbol{\kappa} = \sum_{i=1}^{n} \boldsymbol{B}_{bi} \boldsymbol{w}_i^{(e)}$$

where

$$\boldsymbol{w}_i^{(e)} = \left[\begin{array}{ccc} w_i & \beta_{xi} & \beta_{yi} \end{array} \right]^T.$$

Transverse shear deformations are

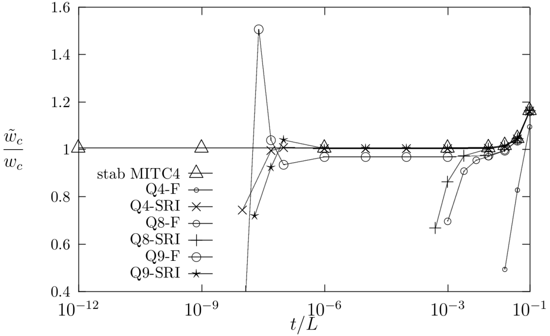
$$\boldsymbol{\gamma} = \left\{ \begin{array}{c} w_{,x} - \beta_x \\ w_{,y} - \beta_y \end{array} \right\} = \sum_{i=1}^{n} \left[\begin{array}{ccc} N_{i,x} & -N_i & 0 \\ N_{i,y} & 0 & -N_i \end{array} \right] \left\{ \begin{array}{c} w_i \\ \beta_{xi} \\ \beta_{yi} \end{array} \right\}, \qquad \boldsymbol{\gamma} = \sum_{i=1}^{n} \boldsymbol{B}_{si} \boldsymbol{w}_i^{(e)}.$$

Locking

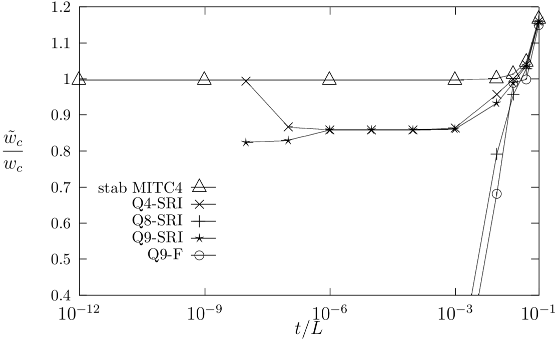
Uniformly loaded clamped plate, 10×10 mesh, (5×5) for biquadratic elements.

F = full integration, SRI = Selective Reduced Integration

$\tilde{w}_c$ = FE-center point displacement, w_c = analytical Kirchhoff model solution.

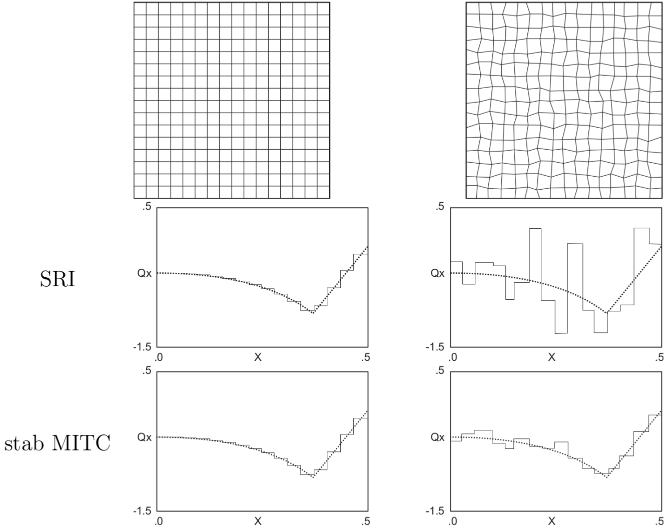


Uniform mesh



Distorted mesh

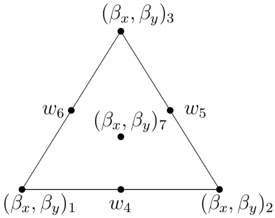
Numerical instability

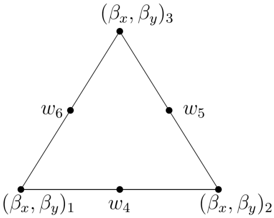


M. Lyly, R. Stenberg, New three and four node plate bending elements, *Rakenteiden Mekaniikka*, vol 27, no 2, ss. 3-29, 1994

Stable and locking-free Reissner-Mindlin plate elements

1. Arnold-Falk triangle: non-conforming iterpolation for w and linear + cubic bubble for rotations.





2. Stabilized MITC elements.

Construction of linear/bilinear MITC elements

For deflection w and rotations β_x, β_y standard C_0-interpolation is used

$$w = \sum_{i=1}^{n} N_i w_i, \qquad \beta_x = \sum_{i=1}^{n} N_i \beta_{xi}, \qquad \beta_y = \sum_{i=1}^{n} N_i \beta_{yi}.$$

Extra rotation interpolation is used for transverse shear computation

$$\beta_x^S = \sum_{i=1}^{n} N_i \beta_{xi}^S, \qquad \beta_y^S = \sum_{i=1}^{n} N_i \beta_{yi}^S.$$

These extra degrees of freedom can be eliminated from the following two constraints.

- Transverse shear strain is constant along the element edge i

$$\gamma_{si} = w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta}^S = \text{constant},$$

- and it is equal to the shear strain computed from the "original" interpolation function at the midpoint of the element edge

$$w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta}^S = w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta}(\zeta = 0).$$

  This condition can also be expressed as an integral

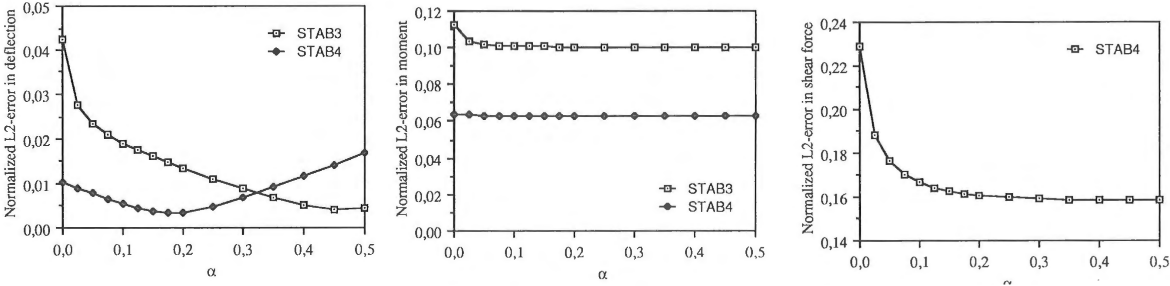
$$\int_{\text{edge } i} \left(w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta}^S \right) ds = \int_{\text{edge } i} \left(w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta} \right) ds \quad \Rightarrow \quad \int_{\text{edge } i} \boldsymbol{s}_i^T \left(\boldsymbol{\beta}^S - \boldsymbol{\beta} \right) ds = 0,$$

Construction of linear/bilinear MITC elements (cont'd)

The shear force has to be computed using the stabilization as

$$Q_x = \frac{kGt}{1+\alpha(h/t)^2}(w_{,x} - \beta_x^S), \qquad Q_y = \frac{kGt}{1+\alpha(h/t)^2}(w_{,y} - \beta_y^S),$$

where α is a positive stabilization parameter and h is a characteristic length of an element, e.g. the length of the longest edge.



M. Lyly, R. Stenberg, New three and four node plate bending elements, *Rakenteiden Mekaniikka*, vol 27, no 2, ss. 3-29, 1994

Generalization, higher-order MITC elements

Constraint

$$\int_{\text{edge } i} \left(w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta}^S \right) ds = \int_{\text{edge } i} \left(w_{,s} - \boldsymbol{s}_i^T \boldsymbol{\beta} \right) ds \quad \Rightarrow \quad \int_{\text{edge } i} \boldsymbol{s}_i^T \left(\boldsymbol{\beta}^S - \boldsymbol{\beta} \right) ds = 0,$$

can be written in a more general form as

$$\int_{\text{edge } i} \boldsymbol{s}_i^T \left(\boldsymbol{\beta}^S - \boldsymbol{\beta} \right) \hat{w} ds, \qquad (1)$$

where $\hat{w}$ is the weight function defined on the element edges, which is one ordee lower degree than the interpolation of deflection w and rotations $\boldsymbol{\beta}$.

Notice that in the case of linearly interpolated element, the weight function w is a consant, i.e. $\hat{w} = 1$.

Elements based on Kirchhoff plate model

Problem: requires C_1-continuity for the interpolation and test functions.

Early trials: Argyris triangle (21 dof), quintic polynomial, Bogner-Fox-Schmit rectangular element with biqubic Hermite interpolation (16 dof). Drawback of these elements are the second derivatives as dofs.

Discrete-Kirchhoff idea: the Kirchoff constraint, i.e. the vanishing transverse shear constraint is imposed only at the integration points.

Shell elements

1. Facet approach: combining flat plate element to 2D-plane stress element.
2. Curved shell elements.

In addition to shear locking, shell elements might exhibit also membrane locking.

Facet shell elements

Usually also so called drilling rotation is added to the membrane element. Two unorthodox approaches:

1. Membrane part: Standard linear/bilinear membrane element combined with non-conforming drilling rotation using the Hughes-Brezzi formulation.
   Bending part: Oñate, Zarate, Flores RM-plate element (conforming deflection, nonconforming rotations).
   R. Kouhia, A novel membrane finite element with drilling rotations, In J.-A. Désidéri, P. Le Tallec, E. Oñate, J. Periaux, and E. Stein, (eds), Numerical Methods in Engineer- ing 96, Proceedings of the Second ECCOMAS Conference on Numerical Methods in Engineering, Paris, France, John Wiley & Sons, Ltd, (1996), 415420.

2. Membrane part: combination of De Veubeke non-conforming triangular plane stress element with conforming drilling rotation using the Hughes-Brezzi formulation.
   Bending part: Arnold-Falk element with Franca-Stenberg modification.

   R. Kouhia, R. Stenberg, A simple linear nonconforming shell element, Fourth International Colloquium on Computation of Shell & Spatial Structures, Chania, Crete, Greece, 2000.

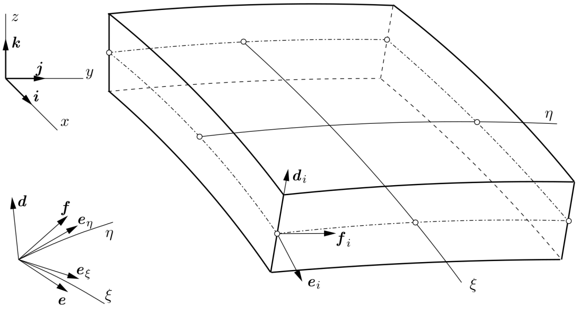
Shell elements - adding drilling rotation

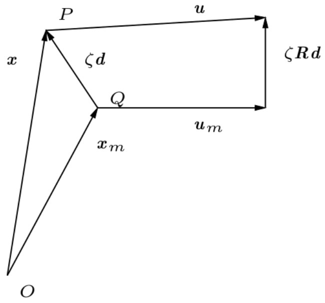
Hughes-Brezzi formulation:

$$\int_\Omega \delta\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}\,\mathrm{d}\Omega + \gamma \int_\Omega (\mathrm{skew}\nabla\delta\boldsymbol{u} - \delta\psi)(\mathrm{skew}\nabla\boldsymbol{u} - \psi)\mathrm{d}\Omega = \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}\Omega$$

The regularizing penalty parameter γ is chosen based on the ellipticity condition. For isotropic materials, a reasonable value is $\gamma =$ (the shear modulus). However, the method is insensitive to the choise of the penalty parameter in the region $0 < \gamma < \mu$ and results for conforming elements suggest to use values $10^{-3} \leq \gamma/\mu \leq 10^{-2}$, which minimizes the condition number of the global stiffness matrix.

Curved isoparametric shell element





Position of an arbitrary point P:

$$\boldsymbol{x}(\xi,\eta,\zeta) = \boldsymbol{x}_{\mathrm{m}}(\xi,\eta) + \zeta \boldsymbol{d}(\xi,\eta),$$

where $\boldsymbol{d}$ is a director vector $\|\boldsymbol{d}\| = 1$. Displacement of P is

$$\boldsymbol{u}(\xi,\eta,\zeta) = \boldsymbol{u}_{\mathrm{m}}(\xi,\eta) + \zeta \boldsymbol{R}(\xi,\eta)\boldsymbol{d}(\xi,\eta),$$

where $\boldsymbol{R}$ describes the rotation of the director vector $\boldsymbol{d}$.

Curved isoparametric shell element (cont'd)

For small rotations, in the linear shell theory, it can be written as

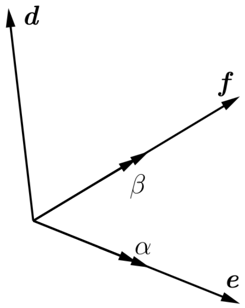
$$\boldsymbol{u} = \boldsymbol{u}_{\rm m} + \zeta(-\alpha \boldsymbol{f} + \beta \boldsymbol{e}),$$

where α and β are rotations wrt the local axis $\boldsymbol{e}$ and $\boldsymbol{f}$, and it is assumed that the rotation about the director vector $\boldsymbol{d}$ vanishes. Geometry is interpolated as

$$\boldsymbol{x}(\xi,\eta,\zeta) = \sum_{i=1}^{n} [N_i(\xi,\eta)\boldsymbol{x}_{mi} + \frac{1}{2}\zeta t_i N_i(\xi,\eta)\boldsymbol{d}_i]$$

Displacements u, v, w are interpolated with same functions (isoparametric element)

$$\boldsymbol{u}(\xi,\eta,\zeta) = \sum_{i=1}^{n} [N_i(\xi,\eta)\boldsymbol{u}_{mi} + \frac{1}{2}\zeta t_i N_i(\xi,\eta)(-\alpha \boldsymbol{f}_i + \beta \boldsymbol{e}_i)],$$



For more details see, R. Kouhia, M. Tuomala Johdatus mekaniikan ja sähkömagnetiikan numeerisiin menetelmiin, Section 13.2.

A good reference is T.J.R. Hughes, The Finite Element Method - Linear Static and Dynamic Finite Element Analysis. Prentice-Hall, 1987, Chapter 6.

Curved isoparametric shell element (cont'd)

Strain-displacement matrix

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{B}\delta\boldsymbol{u}^{(e)} = \sum_{i=1}^{n} \boldsymbol{B}_i \delta\boldsymbol{u}_i^{(e)},$$

where the block related to node i is

$$\boldsymbol{B}_i = \boldsymbol{B}_{1i} + \zeta \boldsymbol{B}_{2i},$$

$$\boldsymbol{B}_{1i} = \begin{bmatrix} N_{i,x} & 0 & 0 & -\frac{1}{2}t_i N_i H_{13} l_2 & \frac{1}{2}t_i N_i H_{13} l_1 \\ 0 & N_{i,y} & 0 & -\frac{1}{2}t_i N_i H_{23} m_2 & \frac{1}{2}t_i N_i H_{23} m_1 \\ 0 & 0 & N_{i,z} & -\frac{1}{2}t_i N_i H_{33} n_2 & \frac{1}{2}t_i N_i H_{33} n_1 \\ N_{i,y} & N_{i,x} & 0 & -\frac{1}{2}t_i N_i (H_{23} l_2 + H_{13} m_2) & \frac{1}{2}t_i N_i (H_{23} l_1 + H_{13} m_1) \\ 0 & N_{i,z} & N_{i,y} & -\frac{1}{2}t_i N_i (H_{33} m_2 + H_{23} n_2) & \frac{1}{2}t_i N_i (H_{33} m_1 + H_{23} n_1) \\ N_{i,z} & 0 & N_{i,x} & -\frac{1}{2}t_i N_i (H_{33} l_2 + H_{13} n_2) & \frac{1}{2}t_i N_i (H_{33} l_1 + H_{13} n_1) \end{bmatrix}$$

$$\boldsymbol{B}_{2i} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2}t_i N_{i,x} l_2 & \frac{1}{2}t_i N_{i,x} l_1 \\ 0 & 0 & 0 & -\frac{1}{2}t_i N_{i,y} m_2 & \frac{1}{2}t_i N_{i,y} m_1 \\ 0 & 0 & 0 & -\frac{1}{2}t_i N_{i,z} n_2 & \frac{1}{2}t_i N_{i,z} n_1 \\ 0 & 0 & 0 & -\frac{1}{2}t_i (N_{i,y} l_2 + N_{i,x} m_2) & \frac{1}{2}t_i (N_{i,y} l_1 + N_{i,x} m_1) \\ 0 & 0 & 0 & -\frac{1}{2}t_i (N_{i,z} m_2 + N_{i,y} n_2) & \frac{1}{2}t_i (N_{i,z} m_1 + N_{i,y} n_1) \\ 0 & 0 & 0 & -\frac{1}{2}t_i (N_{i,z} l_2 + N_{i,x} n_2) & \frac{1}{2}t_i (N_{i,z} l_1 + N_{i,x} n_1) \end{bmatrix}.$$

Curved isoparametric shell element (cont'd)

$$\boldsymbol{K}^{(\mathrm{e})} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{C}\boldsymbol{B} \det \boldsymbol{J} \, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta.$$

The integral can be computed explicitely wrt the ζ coordinate, and

$$\det \boldsymbol{J} \approx t|\boldsymbol{x}_{,\xi} \times \boldsymbol{x}_{,\eta}| \, d\xi d\eta.$$

Strain and stresses are defined in the local coodinate system (x_l, y_l, z_l) at the integration point. $\boldsymbol{C}$ has to be transformed from the local constitutive stiffness matrix, which for isotropic linear elasticity takes the form

$$\boldsymbol{C}_l = \frac{E}{1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & k\frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & k\frac{1-\nu}{2} \end{bmatrix} \qquad \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_l^{\mathrm{T}}\boldsymbol{C}_l\boldsymbol{\varepsilon}_l = \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{T}_\varepsilon^{\mathrm{T}}\boldsymbol{C}_l\boldsymbol{T}_\varepsilon\boldsymbol{\varepsilon},$$

where the transformation matrix $\boldsymbol{T}$ contains the direction cosines.

3D continuum element

Start with the virtual work equation **Total Lagrangian (TL) formulation**

$$-\int_{\Omega_0} \delta \boldsymbol{E} : \boldsymbol{S} \, \mathrm{d}V + \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V + \int_{\partial\Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A - \int_{\Omega_0} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_0 \, \mathrm{d}V = 0$$

Remember, the Green-Lagrange strain (GL) tensor is

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right),$$

and the second Piola-Kirchhoff stress (PK2) $\boldsymbol{S}$ is related to the GL-strain as

$$\boldsymbol{S} = \mathcal{C} : \boldsymbol{E}$$

Linearization of the variation of the GL-strain at state $\boldsymbol{u}_1$ is

$$D\delta\boldsymbol{E} = \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}} D\boldsymbol{F} + D\boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{F}\right) = \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{F}_1 + \Delta\boldsymbol{F}) + (\boldsymbol{F}_1 + \Delta\boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{F}\right).$$

Linearization of the constitutive equations

$$D(\mathcal{C} : \boldsymbol{E}) = \mathcal{C} : \Delta\boldsymbol{E} \quad \text{where} \quad \Delta\boldsymbol{E} = \frac{1}{2}\left(\Delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\Delta\boldsymbol{F}\right).$$

3D continuum element (cont'd)

Linearization of the internal virtual work

$$D\int_{\Omega_0} \delta\boldsymbol{E} : \mathcal{C} : \boldsymbol{E}\,\mathrm{d}V = \int_{\Omega_0} \left(\delta\boldsymbol{E} : D(\mathcal{C} : \boldsymbol{E}) + D\delta\boldsymbol{E} : \mathcal{C} : \boldsymbol{E}_1\right)\mathrm{d}V$$

$$= \int_{\Omega_0} \left[\delta\boldsymbol{E} : \mathcal{C} : \Delta\boldsymbol{E} + \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{F}_1 + \Delta\boldsymbol{F}) + (\boldsymbol{F}_1 + \Delta\boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1\right]\mathrm{d}V$$

$$= \int_{\Omega_0} \left[\delta\boldsymbol{E} : \mathcal{C} : \Delta\boldsymbol{E} + \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\Delta\boldsymbol{F} + \Delta\boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1 + \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1\right]\mathrm{d}V$$

where

$$\delta\boldsymbol{E} = \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\delta\boldsymbol{F}\right) \quad \text{and} \quad \Delta\boldsymbol{E} = \frac{1}{2}\left(\Delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\Delta\boldsymbol{F}\right).$$

$$\int_{\Omega_0} \delta\boldsymbol{E} : \mathcal{C} : \Delta\boldsymbol{E}\,\mathrm{d}V$$ results in the material stiffness matrix,

$$\int_{\Omega_0} \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\Delta\boldsymbol{F} + \Delta\boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1\,\mathrm{d}V$$ results in the geometric stiffness matrix,

$$\int_{\Omega_0} \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1\,\mathrm{d}V$$ results in the internal force vector.

3D continuum element (cont'd)

Interpolation for the displacements and virtual displacements (test functions) and denoting $u = u_1, v = u_2, w = u_3$.

$$u = \sum_{k=1}^{n} N_k u_k, \quad \delta u = \sum_{k=1}^{n} N_k \delta u_k \quad \text{etc.}$$

Putting the strain components into a vector $\hat{\boldsymbol{E}} = (E_{11}, E_{22}, E_{33}, 2E_{12}, 2E_{23}, 2E_{13})^{\mathrm{T}}$, then

$$\hat{\boldsymbol{E}} = \boldsymbol{B}\delta\boldsymbol{q} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n] \begin{pmatrix} \delta\boldsymbol{q}_1 \\ \delta\boldsymbol{q}_2 \\ \vdots \\ \delta\boldsymbol{q} \end{pmatrix}$$

where $\delta\boldsymbol{q}$ contains the virtual displacements as $\delta\boldsymbol{q} = (u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n)^{\mathrm{T}}$ and $\delta\boldsymbol{q}_i$ contains the virtual nodal displacements of a node i: $\delta\boldsymbol{q}_i = (u_i, v_i, w_i)^{\mathrm{T}}$.

3D continuum element (cont'd)

The strain-displacement matrix block for node k is

$$\boldsymbol{B}_k = \begin{bmatrix} F_{11}N_{k,1} & F_{21}N_{k,1} & F_{31}N_{k,i} \\ F_{12}N_{k,2} & F_{22}N_{k,2} & F_{32}N_{k,2} \\ F_{13}N_{k,3} & F_{23}N_{k,3} & F_{33}N_{k,3} \\ F_{11}N_{k,2} + F_{12}N_{k,1} & F_{21}N_{k,2} + F_{22}N_{k,1} & F_{31}N_{k,2} + F_{32}N_{k,1} \\ F_{12}N_{k,3} + F_{13}N_{k,2} & F_{22}N_{k,3} + F_{23}N_{k,2} & F_{32}N_{k,3} + F_{33}N_{k,2} \\ F_{13}N_{k,1} + F_{11}N_{k,3} & F_{23}N_{k,1} + F_{21}N_{k,3} & F_{33}N_{k,1} + F_{31}N_{k,3} \end{bmatrix}$$

Notice the cyclic pattern in indexes.

The material stiffness matrix $\boldsymbol{K}_1^{(\mathrm{e})}$:

$$\int_{\Omega_0^{(\mathrm{e})}} \delta\boldsymbol{E} : \mathcal{C} : \Delta\boldsymbol{E} \,\mathrm{d}V \to \delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}_1^{(\mathrm{e})}\Delta\boldsymbol{q} = \delta\boldsymbol{q}^{\mathrm{T}} \int_{\Omega_0^{(\mathrm{e})}} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}\,\mathrm{d}V\,\Delta\boldsymbol{q}$$

The internal force vector $\boldsymbol{r}^{(\mathrm{e})}$:

$$\int_{\Omega_0^{(\mathrm{e})}} \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}_1 + \boldsymbol{F}_1^{\mathrm{T}}\delta\boldsymbol{F}\right) : \boldsymbol{S}_1 \,\mathrm{d}V \to \delta\boldsymbol{q}^{\mathrm{T}} \int_{\Omega_0^{(\mathrm{e})}} \boldsymbol{B}^{\mathrm{T}}\hat{\boldsymbol{S}}\,\mathrm{d}V.$$

3D continuum element (cont'd)

Goemetric stiffness matrix $\boldsymbol{K}_{\mathrm{g}}^{(\mathrm{e})}$:

$$\int_{\Omega_0}^{(\mathrm{e})} \frac{1}{2}\left(\delta \boldsymbol{F}^{\mathrm{T}} \Delta \boldsymbol{F} + \Delta \boldsymbol{F}^{\mathrm{T}} \delta \boldsymbol{F}\right) : \boldsymbol{S}_1 \,\mathrm{d}V \rightarrow \delta \boldsymbol{q}^{\mathrm{T}} \boldsymbol{K}_{\mathrm{g}}^{(\mathrm{e})} \Delta \boldsymbol{q}$$

3D continuum element - Fortran routine to compute B

```
C====================================================================
C     BM3D — routine to form the strain—sisplacement matrix B for 3—D solid
C--------------------------------------------------------------------
C     MB    = leading dimension of B
C     B     = strain—displacement matrix (output)
C     MPE   = leading dimension of DN array (MPE >= NPE)
C     DN    = derivatives of interpolation functions wrt global coordinates
C     MF    = leading dimension of F
C     F     = deformation gradient or it's increment (TL or UL formulat)
C     NPE   = nodes per element
C====================================================================
```

```
SUBROUTINE BM3D(MB,B,MPE,DN,MF,F,NPE)
IMPLICIT NONE
INTEGER MB,MPE,MF,NPE
REAL*8 B(MB,*),DN(MPE,*),F(MF,*)
INTEGER I,IW,IV,IU
IW = 0
DO I = 1, NPE
   IW = IW + 3
   IV = IW - 1
   IU = IW - 2
   B(1,IU) = F(1,1)*DN(I,1)
   B(1,IV) = F(2,1)*DN(I,1)
   B(1,IW) = F(3,1)*DN(I,1)
   B(2,IU) = F(1,2)*DN(I,2)
   B(2,IV) = F(2,2)*DN(I,2)
   B(2,IW) = F(3,2)*DN(I,2)
   B(3,IU) = F(1,3)*DN(I,3)
   B(3,IV) = F(2,3)*DN(I,3)
   B(3,IW) = F(3,3)*DN(I,3)
   B(4,IU) = F(1,2)*DN(I,1) + F(1,1)*DN(I,2)
   B(4,IV) = F(2,2)*DN(I,1) + F(2,1)*DN(I,2)
   B(4,IW) = F(3,2)*DN(I,1) + F(3,1)*DN(I,2)
   B(5,IU) = F(1,3)*DN(I,2) + F(1,2)*DN(I,3)
   B(5,IV) = F(2,3)*DN(I,2) + F(2,2)*DN(I,3)
   B(5,IW) = F(3,3)*DN(I,2) + F(3,2)*DN(I,3)
   B(6,IU) = F(1,1)*DN(I,3) + F(1,3)*DN(I,1)
   B(6,IV) = F(2,1)*DN(I,3) + F(2,3)*DN(I,1)
   B(6,IW) = F(3,1)*DN(I,3) + F(3,3)*DN(I,1)
 END DO
END
```

3D continuum element - Fortran routine to compute geometric stiffness

```
C===========================================================================
C      GSM3DS — Routine to set up the geometric stiffness matrix of a
C      3D isoparametric solid element
C---------------------------------------------------------------------------
C      Inputs:
C      NPE            = number of nodes in the element
C      DNX, DNY, DNZ  = derivatives of interpolation functions wrt x,y,z
C      SIG            = stress vector (S_x, S_y, S_z, S_xy, S_yz, S_zx)
C      NR             = leading dimension of the ESM array
C      Input/output:
C      ESM            = element stiffness matrix
C===========================================================================
```

```
      SUBROUTINE GSM3DS(NPE,DNX,DNY,DNZ,SIG,NR,ESM)
      IMPLICIT NONE
      INTEGER NPE,NR
      DOUBLE PRECISION DNX(*),DNY(*),DNZ(*),SIG(*),ESM(NR,*)
      INTEGER I,J,I3,I2,I1,J3,J2,J1
      DOUBLE PRECISION SXNX,SYNY,SZNZ,SXYNX,SXYNY
     &      ,SYZNY,SYZNZ,SXZNX,SXZNZ,TMP

      J3 = 0
      DO J = 1, NPE
         J3 = J3 + 3
         J2 = J3 - 1
         J1 = J3 - 2
         SXNX  = SIG(1)*DNX(J)
         SYNY  = SIG(2)*DNY(J)
         SZNZ  = SIG(3)*DNZ(J)
         SXYNX = SIG(4)*DNX(J)
         SXYNY = SIG(4)*DNY(J)
         SYZNY = SIG(5)*DNY(J)
         SYZNZ = SIG(5)*DNZ(J)
         SXZNX = SIG(6)*DNX(J)
         SXZNZ = SIG(6)*DNZ(J)
         I3 = 0
         DO I = 1, NPE
            I3 = I3 + 3
            I2 = I3 - 1
            I1 = I3 - 2
            TMP =  DNX(I)*(SXNX + SXYNY + SXZNZ)
     &           + DNY(I)*(SYNY + SXYNX + SYZNZ)
     &           + DNZ(I)*(SZNZ + SXZNX + SYZNY)
            ESM(I1,J1) = ESM(I1,J1) + TMP
            ESM(I2,J2) = ESM(I2,J2) + TMP
            ESM(I3,J3) = ESM(I3,J3) + TMP
         END DO
      END DO

      END
```