FEM advanced course

Lecture 6 - Truss element with updated Lagrangian formulation, Timoshenko-Reissner beam model

Reijo Kouhia

Tampere University, Structural Mechanics

Algorithm for total Lagrangian formulation of a truss element

Load steps $n = 1, 2, \ldots, n_{\max}$

- Increment load $\boldsymbol{p}_n = \boldsymbol{p}_{n-1} + \Delta\boldsymbol{p}_n$ and set $\boldsymbol{q}_n^{(0)} = \boldsymbol{q}_{n-1}$
- Iterate $i = 0, 1, 2, \ldots, i_{\max}$
  - In each element extract $\boldsymbol{u}$ from $\boldsymbol{q}$ and compute $\boldsymbol{x}_n^{(i)} = \boldsymbol{X} + \boldsymbol{u}_n^{(i)}$ and strains

$$\varepsilon_n^{(i)} = \frac{1}{\ell_0^2}\tfrac{1}{2}(\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{x}}_n^{(i)})^{\mathrm{T}} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{u}}_n^{(i)}$$

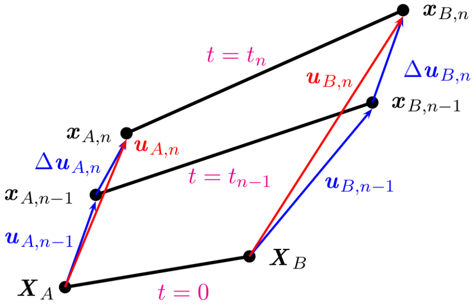
  - Compute internal force vector from element contributions

$$\tilde{\boldsymbol{r}} = \frac{EA_0}{\ell_0}\varepsilon_n^{(i)} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{x}}_n^{(i)} = \frac{N^{(i)}}{\ell_0} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{x}}_n^{(i)}$$

  - Assemble the global stiffness matrix $\boldsymbol{K}_n^{(i)} = \boldsymbol{K}_0(\boldsymbol{X}) + \boldsymbol{K}_u(\boldsymbol{X}, \boldsymbol{u}_n^{(i)}) + \boldsymbol{K}_\sigma(\varepsilon_n^{(i)})$,
  - Compute the global residual force $\boldsymbol{f}_n^{(i)} = \boldsymbol{r}_n^{(i)} - \boldsymbol{p}_n$
  - Solve the linearized system $\boldsymbol{K}_n^{(i)}\delta\boldsymbol{q}_n^{(i)} = \boldsymbol{f}_n^{(i)}$, *notice: δ symbol here means the iterative change!*
  - Update global displacement vector $\boldsymbol{q}_n^{i+1} = \boldsymbol{q}_n^{(i)} - \delta\boldsymbol{q}_n^{(i)}$

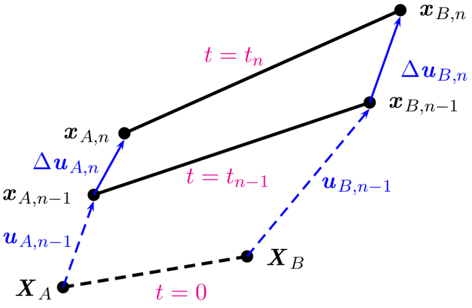
Schematic comparison between TL and UIL formulations

Total Lagrange



$$\boldsymbol{x}_{A,n} = \boldsymbol{X}_A + \boldsymbol{u}_{A,n}$$

Updated incremental Lagrange



$$\boldsymbol{x}_{A,n} = \boldsymbol{x}_{A,n-1} + \Delta\boldsymbol{u}_{A,n}$$

Algorithm for incremental updated Lagrangian formulation of a truss

Load steps $n = 1, 2, \ldots, n_{\max}$

- Increment load $\boldsymbol{p}_n = \boldsymbol{p}_{n-1} + \Delta\boldsymbol{p}_n$ and set $\boldsymbol{q}_n^{(0)} = \boldsymbol{q}_{n-1}$, $\Delta\boldsymbol{q}_n^{(0)} = \boldsymbol{0}$
- Iterate $i = 0, 1, 2, \ldots, i_{\max}$
  - In each element extract $\Delta\boldsymbol{u}$ from $\Delta\boldsymbol{q}$ and compute $\boldsymbol{x}_n^{(i)} = \boldsymbol{x}_{n-1} + \Delta\boldsymbol{u}_n^{(i)}$ and strains

$$\Delta\varepsilon_n^{(i)} = \frac{1}{\ell_0^2}\frac{1}{2}(\tilde{\boldsymbol{x}}_{n-1} + \tilde{\boldsymbol{x}}_n^{(i)})^{\mathrm{T}} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \Delta\tilde{\boldsymbol{u}}_n^{(i)}$$

  - Update strain $\varepsilon_n^{(i)} = \varepsilon_{n-1} + \Delta\varepsilon_n^{(i)}$ and normal force $N_n^{(i)} = EA_0\varepsilon_n$
  - Compute internal force vector from element contributions

$$\tilde{\boldsymbol{r}} = N_n^{(i)} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{x}}_n^{(i)}$$

  - Assemble the global stiffness matrix $\boldsymbol{K}_n^{(i)} = \boldsymbol{K}_0(\boldsymbol{x}_n^{(i)}) + \boldsymbol{K}_\sigma(N_n^{(i)})$,
  - Compute the global residual force $\boldsymbol{f}_n^{(i)} = \boldsymbol{r}_n^{(i)} - \boldsymbol{p}_n$
  - Solve the linearized system $\boldsymbol{K}_n^{(i)}\delta\boldsymbol{q}_n^{(i)} = \boldsymbol{f}_n^{(i)}$, *notice: δ symbol here means the iterative change!*
  - Update global displacement increment vector $\Delta\boldsymbol{q}_n^{i+1} = \Delta\boldsymbol{q}_n^{(i)} - \delta\boldsymbol{q}_n^{(i)}$
- When converged update the nodal cooordinates $\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \Delta\boldsymbol{u}_n$

Timoshenko beam model

Takes the average transverse shear strains into account. It is based on the following two kinematic assumptions:

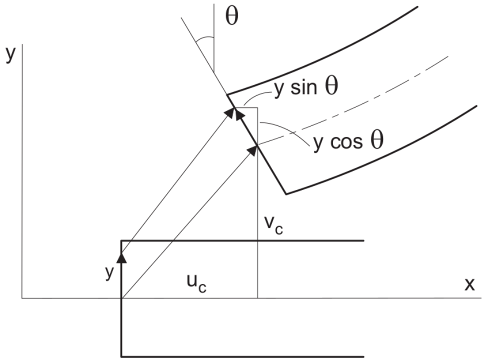
1. fibers normal to the beam's axis in the undeformed state remain straight,
2. these normal fibers do not stretch.

These assumptions can be satisfied by the following displacement field

$$u(x, y) = u_c(x) - y \sin\theta(x),$$
$$v(x, y) = v_c(x) - y(1 - \cos\theta(x)).$$

For linear model $|\theta|, |u/L|, |v/L| \ll 1$ then

$$u(x, y) = u_c(x) - y\theta(x),$$
$$v(x, y) = v_c(x).$$



Virtual work of the Timoshenko beam model

Strains in the Timoshenko beam model are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_c}{\mathrm{d}x} - y\frac{\mathrm{d}\theta}{\mathrm{d}x} = u_c' - y\theta', \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\mathrm{d}v_c}{\mathrm{d}x} - \theta = v_c' - \theta.$$

The virtual work equation is

$$-\int_0^L \int_A (\delta\varepsilon_x \sigma_x + \delta\gamma_{xy}\tau_{xy})\mathrm{d}A\mathrm{d}x + \int_0^L \delta v_c f \mathrm{d}x = 0.$$

Since $\tau = G\gamma$ is constant over the cross-section (without subscripts)

$$\int_A \delta\gamma\tau\,\mathrm{d}A = \delta\gamma\, GA_{\mathrm{s}}\gamma \quad \Rightarrow \quad \int_0^L \delta\gamma Q\,\mathrm{d}x,$$

where A_{s} is the effective shear area of the cross-section.

Virtual work of the Timoshenko beam model (cont'd)

The axial force and bending moment results from ($\sigma = E\varepsilon$)

$$\delta\varepsilon\sigma = (\delta u_c' - y\delta\theta')E(u_c' - y\theta') = \delta u_c' E u_c' - \delta u_c' E y \theta' - \delta\theta' E y u_c' + \delta\theta' E y^2 \theta',$$

and if $\int_A Ey \, \mathrm{d}A = 0$, then

$$\int_0^L \int_A \delta\varepsilon\sigma \, \mathrm{d}A \mathrm{d}x = \int_0^L (\delta u_c' E A u_c' + \delta\theta' E I \theta') \mathrm{d}x.$$

Introducing stress resultants $N = EAu_c'$ and $M = -EI\theta' = EI\kappa$, we get

$$\int_0^L (\delta u_c' N + \delta\kappa M) \mathrm{d}x = \int_0^L (\delta\varepsilon_c N + \delta\kappa M) \mathrm{d}x.$$

Simple Timoshenko beam element

The virtual work equation is now

$$-\int_0^L [\delta u_c' EA u_c' + \delta\theta' EI\theta' + (\delta v_c' - \delta\theta) GA_\mathrm{s}(v_c' - \theta)]\mathrm{d}x + \int_0^L \delta v_c f \,\mathrm{d}x = 0.$$

We have three independent functions to be interpolated u_c, v_c and θ. Notice that the axial and bending deformations decouple if $\int_A Ey \,\mathrm{d}A = 0$.

The simplest possible choice is to use linear C_0-interpolation to all of these functions

$$u_c^{(e)} = N_1(x)u_1^{(e)} + N_2(x)u_2^{(e)}, \quad v_c^{(e)} = N_1(x)v_1^{(e)} + N_2(x)v_2^{(e)}, \quad \theta^{(e)} = N_1(x)\theta_1^{(e)} + N_2(x)\theta_2^{(e)},$$

and also the virtual ones

$$\delta u_c^{(e)} = N_1(x)\delta u_1^{(e)} + N_2(x)\delta u_2^{(e)}, \quad \delta v_c^{(e)} = N_1(x)\delta v_1^{(e)} + N_2(x)\delta v_2^{(e)}, \quad \delta\theta^{(e)} = N_1(x)\delta\theta_1^{(e)} + N_2(x)\delta\theta_2^{(e)}.$$

Simple Timoshenko beam element (cont'd)

Denoting the vector of "strains" and unknown vector we can write the strain-displacement relation

$$\boldsymbol{e} = \begin{pmatrix} \varepsilon \\ \kappa \\ \gamma \end{pmatrix},$$

$$\boldsymbol{q}^{(e)} = \begin{pmatrix} u_1^{(e)} \\ v_1^{(e)} \\ \theta_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ \theta_2^{(e)} \end{pmatrix}$$

$\boldsymbol{e} = \boldsymbol{B}\boldsymbol{q}^{(e)}$,where the strain displacement matrix has the form

$$\boldsymbol{B} = \begin{bmatrix} N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 \\ 0 & 0 & -N_{1,x} & 0 & 0 & -N_{2,x} \\ 0 & N_{1,x} & -N_1 & 0 & N_{2,x} & -N_2 \end{bmatrix}$$

Simple Timoshenko beam element (cont'd)

The material stiffness matrix has the form

$$\boldsymbol{C} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & GA_\mathrm{s} \end{bmatrix}$$

relating the strain vector $\boldsymbol{e}$ to the vector of stress resultants

$$\boldsymbol{\Sigma} = \begin{pmatrix} N \\ M \\ Q \end{pmatrix} = \boldsymbol{C}\boldsymbol{e}.$$

The virtual work equation can now be put in the form

$$\delta\boldsymbol{q}^\mathrm{T}(-\boldsymbol{K}\boldsymbol{q} + \bar{\boldsymbol{p}}) = 0, \quad \text{for all } \delta\boldsymbol{q} \quad \Rightarrow \boldsymbol{K}\boldsymbol{q} = \bar{\boldsymbol{p}}.$$

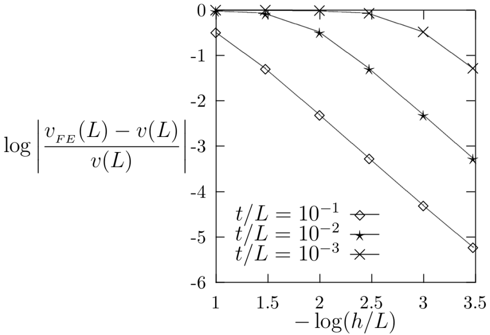
where $\boldsymbol{K}$ and $\bar{\boldsymbol{p}}$ are assembled from the element contributions

$$\boldsymbol{K}^{(e)} = \int_{I^{(e)}} \boldsymbol{B}^\mathrm{T}\boldsymbol{C}\boldsymbol{B}\,\mathrm{d}x, \qquad \bar{\boldsymbol{p}}^{(e)} = \int_{I^{(e)}} [0, N_1, 0, 0, N_2, 0]^\mathrm{T} f\,\mathrm{d}x.$$

Timoshenko beam element - locking

Full integration requires two point integration. However, such approach results an element which locks in the thin beam limit $t \to 0$.

Below is the convergence in tip displacement of a point loaded cantilever beam (also convergence of energy) with three thickness to length values.



Reasons for locking

When equal order interpolation is used spurious shear strains develop

$$\gamma = v_c' - \theta.$$

What to do?

Reasons for locking

When equal order interpolation is used spurious shear strains develop

$$\gamma = v_c' - \theta.$$

What to do?

Should the displacement interpolation be one order higher than rotation interpolation?

Reasons for locking

When equal order interpolation is used spurious shear strains develop

$$\gamma = v_c' - \theta.$$

What to do?

Should the displacement interpolation be one order higher than rotation interpolation?

No, since then shear is linear and curvatute a constant. Remember the equilibrum equation $Q = M'$.

Where we should put the extra effort?

Reasons for locking

When equal order interpolation is used spurious shear strains develop

$$\gamma = v_c' - \theta.$$

What to do?

Should the displacement interpolation be one order higher than rotation interpolation?

No, since then shear is linear and curvatute a constant. Remember the equilibrum equation $Q = M'$.

Where we should put the extra effort?

In rotation since bending moment is more important than shear.

Remedies

Under integration. When linear interpolation is used, use one point quadrature. It will under integrate the terms coming from shear $\gamma = v' - \theta$

$$\boldsymbol{e} = \begin{pmatrix} \varepsilon \\ \kappa \\ \gamma \end{pmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 \\ 0 & 0 & N_{1,x} & 0 & 0 & N_{2,x} \\ 0 & N_{1,x} & N_1 & 0 & N_{2,x} & N_2 \end{bmatrix}$$

It can be viewed also as a projection to a constant, and it is often written as

$$\gamma = \Pi_0(v' - \theta) = v' - \Pi_0\theta,$$

where Π_0 is a projection operator - projection to a constant.

Remedies

Under integration. When linear interpolation is used, use one point quadrature. It will under integrate the terms coming from shear $\gamma = v' - \theta$

$$\boldsymbol{e} = \begin{pmatrix} \varepsilon \\ \kappa \\ \gamma \end{pmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 \\ 0 & 0 & N_{1,x} & 0 & 0 & N_{2,x} \\ 0 & N_{1,x} & N_1 & 0 & N_{2,x} & N_2 \end{bmatrix}$$

It can be viewed also as a projection to a constant, and it is often written as

$$\gamma = \Pi_0(v' - \theta) = v' - \Pi_0\theta,$$

where Π_0 is a projection operator - projection to a constant.

One problem still remains, ill-conditioning of the stiffness matrix when the beam is thin, i.e. $t \to 0$.

Ill-conditioning

Looking the strain energy of the Timoshenko beam model

$$U(v,\theta) = \tfrac{1}{2}\int_0^L [EI(\theta')^2 + GA_{\mathrm{s}}(v'-\theta)^2]\mathrm{d}x$$

Using notations $I = Ar^2$ and $A_{\mathrm{s}} = kA$ we obtain

$$U(v,\theta) = \tfrac{1}{2}\int_0^L EI\left[(\theta')^2 + \frac{k}{2(1+\nu)}\frac{1}{r^2}(v'-\theta)^2\right]\mathrm{d}x.$$

Using dimensionless quantities $\vartheta = v/L$ and $\xi = x/L$, it is obtained

$$U(\vartheta,\theta) = \tfrac{1}{2}\int_0^1 \frac{EI}{L}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2 + \frac{k}{2(1+\nu)}\left(\frac{L}{r}^2\right)\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi} - \theta\right)^2\right]\mathrm{d}x$$

$$= \tfrac{1}{2}\int_0^1 \frac{EI}{L}\left[\hat{\kappa}^2 + \frac{k}{2(1+\nu)}\left(\frac{L}{r}\right)^2\gamma^2\right]\mathrm{d}x, \qquad \hat{\kappa} = L\kappa.$$

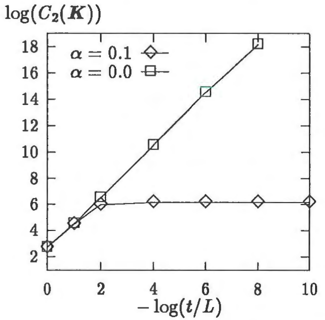
Remedies to ill-conditioning

Since the shear energy has a minor role in thin beam behaviour, we can scale the shear stiffness like

$$GA_{\mathrm{s}} \to \frac{GA_{\mathrm{s}}}{1+\alpha(h^{(e)}/t)^2},$$

where $\alpha > 0$ is a stabilization parameter, $h^{(e)}$ the length of an element and t thickness.

Condition number of the stiffness matrix for 10×10 mesh with stabilized MITC4 Reissner-Mndlin plate elements (300 dofs). Figure from R. Kouhia, Eräitä matala-asteisia laattaelementtejä, *Rakenteiden Mekaniikka*, Vol. 29, Nro 3-4, 1996, pp. 51-68.



Selection of the value for the stabilization

For a beam element with constant cross-section, the follwing reduction of shear stiffness will give exact nodal dispacements for point loadings

$$GA_\mathrm{s} \to \frac{GA_\mathrm{s}}{1 + \dfrac{GA_\mathrm{s}(h^{(e)})^2}{12EI}}.$$

This result was derived by Richard MacNeal in 1978 (*Computers & Structures*, Vol. 8, pp. 175-183). He called it *residual bending flexibility*.

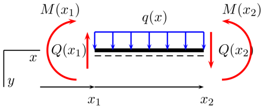
The father of te idea on reducing shear stiffness seems to be Isaac Fried from 1973 (I. Fried, S.K. Yang, *Quarterly of Applied Mathematics*, Vol. 31 pp. 303-312)

There is a close connection between the rotation bubble function and the shear stiffness reduction. This has been proven first time by Juhani Pitkäranta in 1988 (*Numerische Mathematik*, Vol. 53, pp. 237-254)

Extra: Equilibrium equations for the Timoshenko beam model

Derive the equilibrium equations of a beam model, loaded by a vertical force intensity $q(x)$.

The force equilibrium in the vertical direction is



$$Q(x_2) - Q(x_1) + \int_{x_1}^{x_2} q(x)\mathrm{d}x = 0,$$

which can be written as

$$\Big|_{x_1}^{x_2} Q(x) + \int_{x_1}^{x_2} q(x)\,\mathrm{d}x = 0,$$

and furthermore

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}Q}{\mathrm{d}x} + q \right) \mathrm{d}x = 0.$$

Since the values x_1 and x_2 are arbitrary it can be deduced that

$$-\frac{\mathrm{d}Q}{\mathrm{d}x} = q, \quad x \in (0, L). \tag{1}$$

Equilibrium equations for the Timoshenko beam model (cont'd)

The moment equilibrium equation with respect to an arbitrary point x_0 is

$$M(x_1) - M(x_2) + Q(x_2)(x_2 - x_0) - Q(x_1)(x_1 - x_0) + \int_{x_1}^{x_2} q(x)(x - x_0)\,\mathrm{d}x = 0,$$

which can be written as

$$-\Big|_{x_1}^{x_2} M(x) + \Big|_{x_1}^{x_2} Q(x)(x - x_0) + \int_{x_1}^{x_2} q(x)(x - x_0)\,\mathrm{d}x.$$

Proceeding in a similar way as in the previous example gives

$$-\int_{x_1}^{x_2} \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left[Q(x)(x - x_0)\right]\,\mathrm{d}x + \int_{x_1}^{x_2} q(x)(x - x_0)\,\mathrm{d}x = 0,$$

$$-\int_{x_1}^{x_2} \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_1}^{x_2} \left(Q + (x - x_0)\frac{\mathrm{d}Q}{\mathrm{d}x}\right)\mathrm{d}x + \int_{x_1}^{x_2} q(x)(x - x_0)\,\mathrm{d}x = 0$$

$$\int_{x_1}^{x_2} \left(Q - \frac{\mathrm{d}M}{\mathrm{d}x}\right)\mathrm{d}x + \int_{x_1}^{x_2} (x - x_0)\left(\frac{\mathrm{d}Q}{\mathrm{d}x} - q\right)\mathrm{d}x = 0.$$

Due to the vertical force equilibrium equation (1) the last integral vanishes and the moment equilibrium equations results in

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x}, \quad \text{from which} \quad -\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = q.$$

Derivation of the virtual work equation from the equilibrium equations

The principle of virtual work (PVW) is equivalent to equilibrum equations and force boundary conditions.

Equilibrium equations:

$$Q' + q = 0 \qquad \cdot\, \delta v \qquad \int_0^L$$

$$Q - M' = 0 \qquad \cdot\, \delta\theta \qquad \int_0^L$$

Boundary conditions, assume now that displacement v and rotation θ are given at $x = 0$ and force boundary conditions:

$$F - Q(L) = 0 \qquad \cdot\, \delta v(L)$$

$$M(L) - M_L = 0 \qquad \cdot\, \delta\theta(L)$$

$$\int_0^L [(Q' + q)\delta v + (Q - M')\delta\theta]\mathrm{d}x + [F - Q(L)]\delta v(L) + [M(L) - M_L]\delta\theta(L) = 0$$

Derivation of VW equation (cont'd)

$$\int_0^L [(Q' + q)\delta v + (Q - M')\delta\theta]\mathrm{d}x + [F - Q(L)]\delta v(L) + [M(L) - M_L]\delta\theta(L) = 0$$

Integration by parts gives

$$\Big|_0^L Q\delta v - \int_0^L Q\delta v' \,\mathrm{d}x + \int_0^L q\delta v \,\mathrm{d}x + \int_0^L Q\delta\theta \,\mathrm{d}x - \Big|_0^L M\delta\theta + \int_0^L M\delta\theta' \,\mathrm{d}x +$$
$$+ [F - Q(L)]\delta v(L) + [M(L) - M_L]\delta\theta(L) = 0$$

Since v and θ are prescribed at $x = 0$ then $\delta v(0) = 0$ and $\delta\theta(0)$, after rearrangements

$$-\int_0^L [Q(\delta v' - \delta\theta) - M\delta\theta'] \,\mathrm{d}x + \int_0^L q\delta v \,\mathrm{d}x + F\delta v(L) - M_L\delta\theta(L) = 0$$

Remember that $\delta v' - \delta\theta = \delta\gamma$ and $-\delta\theta' = \delta\kappa$, thus

$$-\int_0^L (Q\delta\gamma + M\delta\kappa) \,\mathrm{d}x + \int_0^L q\delta v \,\mathrm{d}x + F\delta v(L) - M_L\delta\theta(L) = 0$$

Next

Exercises on Thursday at 2 PM.

Coding updated Lagrangian truss element or linear Timoshenko beam element?

Next lecture, non-linear Reissner beam element. Path following algorithms.