FEM advanced course

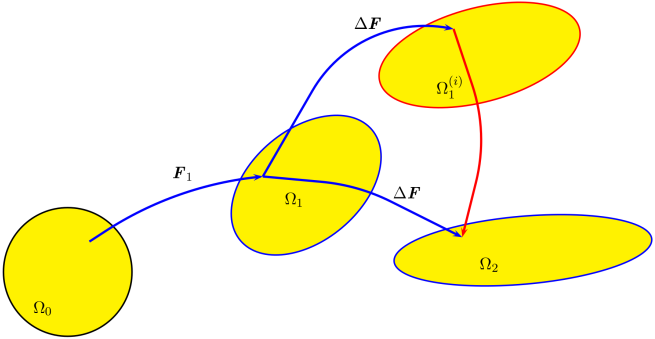
Lecture 5 - Updated Lagrangian formulations, truss element

Reijo Kouhia

Tampere University, Structural Mechanics

Incremental descriptions

1. Total Lagrangian formulation. Reference configuration is the initial configuration Ω_0.
2. Updated Lagrangian formulation
   1. Reference configuration is the last equilibrium state Ω_1. **Incrementally updated Lagrangian**.
   2. Reference configuration is the state from the last iterate $\Omega_1^{(i)}$, weather or not it is in equilibrium. **Updated Lagrangian**.
3. Eulerian formulation. Reference to the current state Ω_2.



Principle of virtual work (PVW)

Slight change in notation - the left subscript in tensor quantities indicates the reference state. **Total Lagrangian (TL) formulation**

$$-\int_{\Omega_0} \delta_0 \boldsymbol{E} : {}_0\boldsymbol{S}\, \mathrm{d}V_0 + \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}}\, \mathrm{d}V_0 + \int_{\partial\Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}}\, \mathrm{d}A_0 - \int_{\Omega_0} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_0\, \mathrm{d}V_0 = 0$$

Incremental Updated Lagrangian (IL) formulation

$$-\int_{\Omega_1} \delta_1 \boldsymbol{E} : {}_1\boldsymbol{S}\, \mathrm{d}V_1 + \int_{\Omega_1} \delta \boldsymbol{u} \cdot \rho_1 \bar{\boldsymbol{b}}\, \mathrm{d}V_1 + \int_{\partial\Omega_{t1}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}}\, \mathrm{d}A_1 - \int_{\Omega_1} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_1\, \mathrm{d}V_1 = 0$$

Eulerian formulation - Updated Lagrangian formulation (UL)

$$-\int_{\Omega_2} \delta_2 \boldsymbol{e} : {}_2\boldsymbol{\sigma}\, \mathrm{d}V_2 + \int_{\Omega_2} \delta \boldsymbol{u} \cdot \rho_2 \bar{\boldsymbol{b}}\, \mathrm{d}V_2 + \int_{\partial\Omega_{t2}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}}\, \mathrm{d}A_2 - \int_{\Omega_2} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_2\, \mathrm{d}V_2 = 0$$

Variation or linearization of a spatial field is formally equivalent to the Lie time derivative.

Variation of the Almansi strain tensor

Variation of the Eulerian Almansi strain tensor:

1. Apply the pull back operation to obtain a material field.

$$ {}^2_0\boldsymbol{F}^T({}^2_2\boldsymbol{e}){}^2_0\boldsymbol{F} = {}^2_0\boldsymbol{E}, \quad \text{in the sequel} \quad \boldsymbol{F}^T\boldsymbol{e}\boldsymbol{F} = \boldsymbol{E}. $$

2. Take the variation of the material Green-Lagrange tensor

$$ \delta\boldsymbol{E} = \tfrac{1}{2}(\delta\boldsymbol{H}^T\boldsymbol{F} + \boldsymbol{F}^T\delta\boldsymbol{H}) = \mathrm{sym}\delta\boldsymbol{H}^T\boldsymbol{F} $$

3. Apply the push forward operation to obtain the spatial field:

$$ \boldsymbol{F}^{-T}\delta\boldsymbol{E}\boldsymbol{F}^{-1} = \boldsymbol{F}^{-T}\tfrac{1}{2}(\delta\boldsymbol{H}^T\boldsymbol{F} + \boldsymbol{F}^T\delta\boldsymbol{H})\boldsymbol{F}^{-1} = \boldsymbol{F}^{-T}\tfrac{1}{2}[(\mathrm{Grad}\delta\boldsymbol{u})^T\boldsymbol{F} + \boldsymbol{F}^T\mathrm{Grad}\delta\boldsymbol{u}]\boldsymbol{F}^{-1} $$

Notice that the spatial gradient $\mathrm{grad}\delta\boldsymbol{u} = \mathrm{Grad}\delta\boldsymbol{u}\,\boldsymbol{F}^{-1}$, thus

$$ \boldsymbol{F}^{-T}\tfrac{1}{2}[(\mathrm{Grad}\delta\boldsymbol{u})^T\boldsymbol{F} + \boldsymbol{F}^T\mathrm{Grad}\delta\boldsymbol{u}]\boldsymbol{F}^{-1} = \tfrac{1}{2}[(\mathrm{grad}\delta\boldsymbol{u})^T + \mathrm{grad}\delta\boldsymbol{u}] = \delta\boldsymbol{e}. $$

Internal virtual work

It has to be equivalent

$$-\int_{\Omega_0} \delta_0^2\boldsymbol{E} : {}_0^2\boldsymbol{S}\,\mathrm{d}V_0 = -\int_{\Omega_2} \delta_2^2\boldsymbol{e} : {}_2^2\boldsymbol{\sigma}\,\mathrm{d}V_2$$

Taking into account equations

$${}_0^2\boldsymbol{S} = J({}_0^2\boldsymbol{F}^{-1}){}_2^2\boldsymbol{\sigma}{}_0^2\boldsymbol{F}^{-T} \quad \delta_0^2\boldsymbol{E} = {}_0^2\boldsymbol{F}^T\delta_2^2\boldsymbol{e}{}_0^2\boldsymbol{F},$$

we get

$$-\int_{\Omega_0} \boldsymbol{F}^T\delta\boldsymbol{e}\boldsymbol{F} : \boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}J\mathrm{d}V_0 = -\int_{\Omega_2} \delta\boldsymbol{e} : \boldsymbol{\sigma}\mathrm{d}V_2.$$

Internal virtual work (cont'd)

Here $\boldsymbol{F} = {}^2_0\boldsymbol{F}$ etc.

Let us look a little bit closer the term $\boldsymbol{F}^T\delta\boldsymbol{e}\boldsymbol{F} : \boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$. It is easy to simplify in the index form

$$\delta E_{KL} = F_{pK}\delta e_{pq}F_{qL}, \qquad S_{KL} = JF^{-1}_{Km}\sigma_{mn}F^{-1}_{Ln},$$

the inner product is then

$$\begin{aligned}\delta\boldsymbol{E} : \boldsymbol{S} &= \delta E_{KL}S_{KL} = JF_{pK}\delta e_{pq}F_{qL}F^{-1}_{Km}\sigma_{mn}F^{-1}_{Ln} = J\delta_{pm}\delta_{qn}\delta e_{pq}\sigma_{mn} \\ &= J\delta e_{mn}\sigma_{mn} = J\delta\boldsymbol{e} : \boldsymbol{\sigma}\end{aligned}$$

Linearization of the internal virtual work

In the total Lagrangian formulation

$$-\int_{\Omega_0} \delta \boldsymbol{E} : \boldsymbol{S} \, \mathrm{d}V \tag{1}$$

Assuming constitutive equation in the form $\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ and we are in the displaced state $\boldsymbol{u}_1$ and we try to solve the increment to obtain $\boldsymbol{u}_2 = \boldsymbol{u}_1 + \Delta \boldsymbol{u}$. At the configuration 1 stresses are denoted as $\boldsymbol{S}_1$ and then

$$\boldsymbol{S}_2 = \boldsymbol{S}_1 + \Delta \boldsymbol{S} = \boldsymbol{S}_1 + \mathbb{C} : \Delta \boldsymbol{E},$$

substituting it and $\delta \boldsymbol{E}$, $\Delta \boldsymbol{E}$ and $\boldsymbol{F}_2 = \boldsymbol{F}_1 + \Delta \boldsymbol{F} = \boldsymbol{F}_1 + \Delta \boldsymbol{H}$ into the internal VW-expression (1) gives

$$-\int_{\Omega_0} \tfrac{1}{2}[\delta \boldsymbol{H}^T(\boldsymbol{F}_1 + \Delta \boldsymbol{H}) + (\boldsymbol{F}_1^T + \Delta \boldsymbol{H}^T)\delta \boldsymbol{H}] : (\boldsymbol{S}_1 + \mathbb{C} : \tfrac{1}{2}[\Delta \boldsymbol{H}^T(\boldsymbol{F}_1 + \Delta \boldsymbol{H}) + (\boldsymbol{F}_1^T + \Delta \boldsymbol{H})\Delta \boldsymbol{H}]) \, \mathrm{d}V \tag{2}$$

Linearization of the internal virtual work - updated formulation

Since

$$\delta \boldsymbol{e} = \tfrac{1}{2}[(\mathrm{grad}\delta\boldsymbol{u})^T + \mathrm{grad}\delta\boldsymbol{u}] = \tfrac{1}{2}(\delta\boldsymbol{h}^T + \delta\boldsymbol{h}) = \mathrm{sym}\delta\boldsymbol{h},$$

also

$$\Delta \boldsymbol{e} = \tfrac{1}{2}[(\mathrm{grad}\Delta\boldsymbol{u})^T + \mathrm{grad}\Delta\boldsymbol{u}] = \tfrac{1}{2}(\Delta\boldsymbol{h}^T + \Delta\boldsymbol{h}) = \mathrm{sym}\Delta\boldsymbol{h},$$

Starting from

$$\Delta\left(-\int_{\Omega_0}\delta\boldsymbol{E}:\boldsymbol{S}\,\mathrm{dV}\right) = -\int_{\Omega_0}[\delta\boldsymbol{E}:\Delta\boldsymbol{S} + \Delta(\delta\boldsymbol{E}):\boldsymbol{S}]\,\mathrm{d}V, \tag{3}$$

applying push forward (contravariant tensor) to the term $\Delta\boldsymbol{S} = \mathbb{C}:\Delta\boldsymbol{E}$, and $\Delta\boldsymbol{E} = \mathrm{sym}(\boldsymbol{F}^T\mathrm{Grad}\Delta\boldsymbol{u})$ we get

$$\Delta\boldsymbol{\tau} = \boldsymbol{F}\Delta\boldsymbol{S}\boldsymbol{F}^T = \boldsymbol{F}(\mathbb{C}:\boldsymbol{F}^T\mathrm{Grad}\Delta\boldsymbol{u})\boldsymbol{F}^T = \boldsymbol{F}(\mathbb{C}:\boldsymbol{F}^T\mathrm{grad}\Delta\boldsymbol{u}\boldsymbol{F})\boldsymbol{F}^T,$$

where the minor symmetries of $\mathbb{C}$ is taken into account.

Linearization of the internal virtual work - updated formulation (cont'd)

The stress increment

$$\Delta\boldsymbol{\tau} = \boldsymbol{F}\Delta\boldsymbol{S}\boldsymbol{F}^T = \boldsymbol{F}(\mathbb{C} : \boldsymbol{F}^T \mathrm{grad}\Delta\boldsymbol{u}\boldsymbol{F})\boldsymbol{F}^T,$$

can be put in the form

$$\Delta\boldsymbol{\tau} = J\mathbb{c} : \mathrm{grad}\Delta\boldsymbol{u}, \tag{4}$$

where the spatial constitutive tensor $\mathbb{c}$ is

$$c_{ijkl} = J^{-1}F_{iM}F_{jN}F_{kP}F_{lQ}C_{MNPQ}.$$

Linearization of the internal virtual work - updated formulation (cont'd)

The second term in

$$-\int_{\Omega_0}\left[\delta\boldsymbol{E}:\Delta\boldsymbol{S}+\Delta(\delta\boldsymbol{E}):\boldsymbol{S}\right]\mathrm{d}V$$

is easy, just applying push forward to covariant tensor $\Delta(\delta\boldsymbol{E})$ and contravariant $\boldsymbol{S}$ and noticing that $\Delta\delta\boldsymbol{e}=\mathrm{sym}[(\mathrm{grad}\delta\boldsymbol{u})^T\mathrm{grad}\Delta\boldsymbol{u}]$

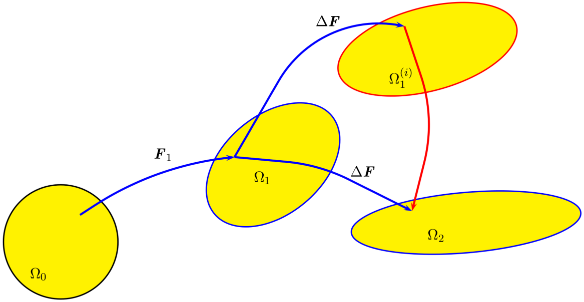
$$\begin{aligned}-\int_{\Omega_0}\boldsymbol{F}^{-T}\Delta(\delta\boldsymbol{E})\boldsymbol{F}^{-1}:\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T\,\mathrm{d}V=-\int_{\Omega_0}\Delta(\delta\boldsymbol{e}):\boldsymbol{\tau}\,\mathrm{d}V&=-\int_{\Omega_2}\Delta(\delta\boldsymbol{e}):\boldsymbol{\sigma}\,\mathrm{d}V_2\\&=-\int_{\Omega_2}\mathrm{grad}\delta\boldsymbol{u}:\mathrm{grad}\Delta\boldsymbol{u}\boldsymbol{\sigma}\mathrm{d}V_2\end{aligned}$$

and taking (4) into account, the linearization of the internal virtual work in the spatial description is

$$-\int_{\Omega_2}\left(\mathrm{grad}\delta\boldsymbol{u}:\mathbb{c}:\mathrm{grad}\Delta\boldsymbol{u}+\mathrm{grad}\delta\boldsymbol{u}:\mathrm{grad}\Delta\boldsymbol{u}\boldsymbol{\sigma}\right)\mathrm{d}V_2.$$

Linearization of the internal virtual work - incremental updated formulation

Reference state the configuration Ω_1.



Notice that

$$ {}_0^2\boldsymbol{F} = \Delta_1^2\boldsymbol{F}({}_0^1\boldsymbol{F}) $$

We need

$$ {}_1^2\boldsymbol{S} = {}_1^1\boldsymbol{\sigma} + \Delta_1^2\boldsymbol{S} = {}_1^1\boldsymbol{\sigma} + {}_1^2\mathbb{C} : \Delta_1^2\boldsymbol{E}, $$

Incremental updated formulation (cont'd)

Incremental updated Lagrangian approach is quite similar to the TL approach, however, some modifications needed.

1. After every step the stresses ${}^2_1\boldsymbol{S}$ has to be transformed to Cauchy stresses
$$ {}^2_2\boldsymbol{\sigma} = J^{-1}\Delta^2_1\boldsymbol{F}^2_1\boldsymbol{S}^2_1\Delta\boldsymbol{F}^T $$
2. Constitutive operator need to be transformed (similar to Eulerian approach), but with respect to the configuration 1
$$ {}^2_1C_{ijkl} = (J^{-1})({}^1_0F_{iM})({}^1_0F_{jN})({}^1_0F_{kP})({}^1_0F_{lQ}){}^2_0C_{MNPQ}, $$
where $J = \det({}^1_0\boldsymbol{F})$.

Truss element in 1, 2 and 3-D

Professor emeritus Steen Krenk has presented an elegant formulation for a truss element in his book *Non-linear modeling and analysis of solids and structures*, Cambridge University Press 2009.

Two nodes A and B having coordinates $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ in the current state. Initial positions are $\boldsymbol{X}_A$ and $\boldsymbol{X}_B$. Vectors connection points A and B are

$$\boldsymbol{X} = \boldsymbol{X}_B - \boldsymbol{X}_A, \quad \text{and} \quad \boldsymbol{x} = \boldsymbol{x}_B - \boldsymbol{x}_A,$$

and the length of an element in the initial ℓ_0 and deformed configurations ℓ are

$$\ell_0^2 = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}, \quad \text{and} \quad \ell^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}.$$

Displacements at nodes A and B are denoted as $\boldsymbol{u}_A$ and $\boldsymbol{u}_B$ respectively and $\boldsymbol{u} = \boldsymbol{u}_B - \boldsymbol{u}_A$, then $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}$ and

$$\ell^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = (\boldsymbol{X} + \boldsymbol{u})^{\mathrm{T}}(\boldsymbol{X} + \boldsymbol{u}).$$

The Green-Lagrange strain is easily computed as (now ε is used for the GL strain)

$$\varepsilon = \frac{\ell^2 - \ell_0^2}{2\ell_0^2} = \frac{1}{\ell_0^2}\left(\boldsymbol{X} + \tfrac{1}{2}\boldsymbol{u}\right)^{\mathrm{T}}\boldsymbol{u} = \frac{1}{\ell_0^2}\frac{1}{2}\left(\boldsymbol{X} + \boldsymbol{X} + \boldsymbol{u}\right)^{\mathrm{T}}\boldsymbol{u} = \frac{1}{\ell_0^2}\frac{1}{2}\left(\boldsymbol{X} + \boldsymbol{x}\right)^{\mathrm{T}}\boldsymbol{u} = \frac{1}{\ell_0^2}\boldsymbol{x}_{1/2}^{\mathrm{T}}\boldsymbol{u}.$$

Truss element in 1, 2 and 3-D (cont'd)

The virtual strain is

$$\delta\varepsilon = \frac{1}{\ell_0^2}\left(\boldsymbol{X}+\boldsymbol{u}\right)^{\mathrm{T}}\delta\boldsymbol{u} = \frac{1}{\ell_0^2}\boldsymbol{x}^{\mathrm{T}}\delta\boldsymbol{u},$$

and the virtual work equation takes the form (element contribution)

$$\begin{aligned}\delta W &= -\int_0^{\ell_0} N\delta\varepsilon\,\mathrm{d}s + \boldsymbol{p}_A\delta\boldsymbol{u}_A + \boldsymbol{p}_B\delta\boldsymbol{u}_B \\ &= -\int_0^{\ell_0}\frac{1}{\ell_0^2}(N\boldsymbol{x}^{\mathrm{T}})(\delta\boldsymbol{u}_B-\delta\boldsymbol{u}_A)\,\mathrm{d}s + \boldsymbol{p}_A\delta\boldsymbol{u}_A + \boldsymbol{p}_B\delta\boldsymbol{u}_B \\ &= \delta\boldsymbol{u}_A^{\mathrm{T}}\left(\int_0^{\ell_0}\frac{1}{\ell_0^2}(N\boldsymbol{x})\mathrm{d}s + \boldsymbol{p}_A\right) + \delta\boldsymbol{u}_B^{\mathrm{T}}\left(-\int_0^{\ell_0}\frac{1}{\ell_0^2}(N\boldsymbol{x})\mathrm{d}s + \boldsymbol{p}_B\right) = 0\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{p}_A = -\frac{1}{\ell_0}N\boldsymbol{x}, \quad \boldsymbol{p}_B = \frac{1}{\ell_0}N\boldsymbol{x}$$

and the internal forces are

$$\boldsymbol{r}_A = -\frac{1}{\ell_0}N\boldsymbol{x}, \quad \boldsymbol{r}_B = \frac{1}{\ell_0}N\boldsymbol{x}$$

Truss element in 1, 2 and 3-D (cont'd)

Assuming the constitutive equation

$$N = EA_0\varepsilon$$

then the internal forces are

$$\boldsymbol{r}_A = -EA_0\varepsilon\frac{\boldsymbol{x}}{\ell_0}, \quad \boldsymbol{r}_B = EA_0\varepsilon\frac{\boldsymbol{x}}{\ell_0}.$$

Notation:

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X}_A \\ \boldsymbol{X}_B \end{pmatrix}, \quad \tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x}_A \\ \boldsymbol{x}_B \end{pmatrix}, \quad \tilde{\boldsymbol{u}} = \begin{pmatrix} \boldsymbol{u}_A \\ \boldsymbol{u}_B \end{pmatrix}, \quad \tilde{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{r}_A \\ \boldsymbol{r}_B \end{pmatrix}.$$

The initial element length is

$$\ell_0^2 = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \tilde{\boldsymbol{X}}^{\mathrm{T}}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{X}},$$

and the Green-Lagrange strain can be written as

$$\varepsilon = \frac{1}{\ell_0^2}\tfrac{1}{2}(\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{x}})^{\mathrm{T}}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{u}},$$

and the virtual strain takes the form

$$\delta\varepsilon = \frac{1}{\ell_0^2}\tilde{\boldsymbol{x}}^{\mathrm{T}}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\delta\tilde{\boldsymbol{u}}.$$

Truss element in 1, 2 and 3-D (cont'd)

The internal forces have the expressions

$$\tilde{\boldsymbol{r}} = \frac{N}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{x}} = \frac{EA_0\varepsilon}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{x}}.$$

Notice that the strain and internal force are non-linear functions of displacements.

Linearization.

$$\Delta\tilde{\boldsymbol{r}} = \frac{\Delta N}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{x}} + \frac{N}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\Delta\tilde{\boldsymbol{x}} = \frac{EA_0\Delta\varepsilon}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\tilde{\boldsymbol{x}} + \frac{N}{\ell_0}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\Delta\tilde{\boldsymbol{u}}.$$

From the previous slide

$$\Delta\varepsilon = \frac{1}{\ell_0^2}\tilde{\boldsymbol{x}}^{\mathrm{T}}\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\Delta\tilde{\boldsymbol{u}}.$$

Truss element in 1, 2 and 3-D (cont'd)

Inserting $\Delta\varepsilon$ to the increment of the internal force we get

$$\Delta\tilde{\boldsymbol{r}} = \left(\frac{EA_0}{\ell_0^3} \begin{pmatrix} \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} & -\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \\ -\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} & \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \end{pmatrix} + \frac{N}{\ell_0} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \right) \Delta\tilde{\boldsymbol{u}} = \boldsymbol{K}\Delta\tilde{\boldsymbol{u}}.$$

The Jacobian matrix $\boldsymbol{K}$ is often called also as the tangent stiffness matrix, and it can be decomposed into three parts

$$\boldsymbol{K}_0 = \frac{EA_0}{\ell_0^3} \begin{pmatrix} \boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} & -\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \\ -\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} & \boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \end{pmatrix},$$

$$\boldsymbol{K}_u = \frac{EA_0}{\ell_0^3} \begin{pmatrix} \boldsymbol{X}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{X}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} & -(\boldsymbol{X}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{X}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\ -(\boldsymbol{X}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{X}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) & \boldsymbol{X}\boldsymbol{u}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{X}^{\mathrm{T}} + \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \end{pmatrix},$$

$$\boldsymbol{K}_\sigma = \frac{N}{\ell_0} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}.$$

Algorithm for total Lagrangian formulation of a truss element

Load steps $n = 1, 2, \ldots, n_{\max}$

- Increment load $\boldsymbol{p}_n = \boldsymbol{p}_{n-1} + \Delta\boldsymbol{p}_n$ and set $\boldsymbol{q}_n^{(0)} = \boldsymbol{q}_{n-1}$
- Iterate $i = 0, 1, 2, \ldots, i_{\max}$
  - In each element extract $\boldsymbol{u}$ from $\boldsymbol{q}$ and compute $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}$ and strains

$$\varepsilon_n^{(i)} = \frac{1}{\ell_0^2}\frac{1}{2}(\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{x}})^{\mathrm{T}} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{u}}_n^{(i)}$$

  - Compute internal force vector from element contributions

$$\tilde{\boldsymbol{r}} = \frac{EA_0}{\ell_0}\varepsilon_n^{(i)} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{pmatrix} \tilde{\boldsymbol{x}}_n^{(i)}$$

  - Assemble the global stiffness matrix $\boldsymbol{K}_n^{(i)} = \boldsymbol{K}_0(\boldsymbol{X}) + \boldsymbol{K}_u(\boldsymbol{X}, \boldsymbol{u}_n^{(i)}) + \boldsymbol{K}_\sigma(\varepsilon_n^{(i)})$,
  - Compute the global residual force $\boldsymbol{f}_n^{(i)} = \boldsymbol{r}_n^{(i)} - \boldsymbol{p}_n$
  - Solve the linearized system $\boldsymbol{K}_n^{(i)}\delta\boldsymbol{q}_n^{(i)} = \boldsymbol{f}_n^{(i)}$, *notice: δ symbol here means the iterative change!*
  - Update global displacement vector $\boldsymbol{q}_n^{i+1} = \boldsymbol{q}_n^{(i)} - \delta\boldsymbol{q}_n^{(i)}$

Next

Exercises on Thursday.
Coding 1,2 and 3-D (same code) total Lagrangian truss element.

Next lecture, truss element with updated Lagrangian formulation, 2-D Reissner beam.