FEM advanced course

Lecture 4 - Objective time rates, total- and updated Lagrangian formulations

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Objectivity

Motivation: *Material properties must be invariant under changes of observers.*

Observer in the Euclidean space is equipped to measure

1. *relative positions* of points in space, and
2. *instants of time*.

An event is noticed by an observer in terms of position $\boldsymbol{x}$ and time t.

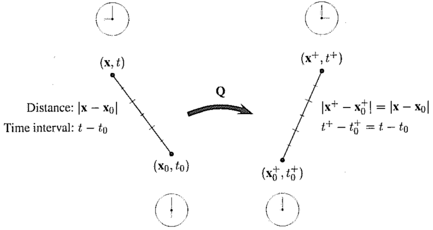


Figure 5.1 from G.A. Holzapfel, *Nonlinear Solid Mechanics*, John Wiley & Sons, 2000.

A spatial mapping satisfying these requirements can be represented by transformation

$$\boldsymbol{x}^+ - \boldsymbol{x}_0 = \boldsymbol{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0),$$

with a proper orthogonal tensor $\boldsymbol{Q}(t)$. The transformation can be written as

$$\boldsymbol{x}^+ = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}, \quad t^+ = t + t_0^+ - t_0.$$

Change of observer

A frame is a rigid reference system from which we observe a a position x of a certain object at a certain time t. A frame may therefore be called an observer.

Same event is recorded

- In frame $\mathcal{F}$ we record $(\boldsymbol{x}, t)$.
- In frame $\mathcal{F}^+$ we record $(\boldsymbol{x}^+, t^+)$. For simplicity assume $t^+ = t$.

In frame $\mathcal{F}$ and frame $\mathcal{F}^+$, the coordinates of the same particle are related as

$$\boldsymbol{x}^+ = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t),$$

where $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$ and $\det \boldsymbol{Q} = 1$.

Objective tensors

Tensors that transform in a similar manner when the frame is changed as when the coordinate system is changed are called *objective tensors*.

Definition of objective tensors

$f^+ = f$	objective scalar
$\boldsymbol{b}^+ = \boldsymbol{Q}\boldsymbol{b}$	objective vector
$\boldsymbol{T}^+ = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T$	objective second-order tensor

Definition of invariant objective tensors

$f^+ = f$	invariant objective scalar
$\boldsymbol{b}^+ = \boldsymbol{b}$	invariant objective vector
$\boldsymbol{T}^+ = \boldsymbol{T}$	invariant objective second-order tensor

Objectivity of deformation gradient $\boldsymbol{F}$?

- In frame $\mathcal{F}$ we have motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$, and
- in frame $\mathcal{F}^+$ we have motion $\boldsymbol{x}^+ = \boldsymbol{\varphi}^+(\boldsymbol{X}, t)$.

Now the deformation gradients recorded by the two frames are

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} \quad \text{and} \quad \boldsymbol{F}^+ = \frac{\partial \boldsymbol{\varphi}^+}{\partial \boldsymbol{X}}$$

and the motions are related as $\boldsymbol{x}^+ = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}$, then

$$\boldsymbol{F}^+ = \frac{\partial \boldsymbol{x}^+}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{x}^+}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{Q}\boldsymbol{F}.$$

Deformation gradient is a two-point tensor having one base $(\boldsymbol{E}_K)$ in the material coordinate system and one in the spatial coordinate system $(\boldsymbol{e}_m)$

$$\boldsymbol{F} = \frac{\partial \varphi_i}{\partial X_J} \boldsymbol{e}_i \otimes \boldsymbol{E}_J,$$

and thus transforms like a vector and can be considered as objective.

Objectivity of some quantities

- The Jacobian $J = \det \boldsymbol{F}$ is objective.
- $\boldsymbol{C}, \boldsymbol{E}, \boldsymbol{U}$ are invariant objective.
- The rate of deformation tensor $\boldsymbol{d}$ is objective.
- The spin tensor $\boldsymbol{w}$ is **not objective**.
- The traction vector $\boldsymbol{t}$ is assumed to be objective thus the Cauchy stress $\boldsymbol{\sigma}$ is objective.
- The PK2 stress tensor $\boldsymbol{S}$ is invariant objective.
- The material time rate of the GL strain tensor $\dot{\boldsymbol{E}}$ is invariant objective.

The material time derivatice of the Cauchy stress tensor

The Cauchy stress tensor $\boldsymbol{\sigma}$ is objective, thus $\boldsymbol{\sigma}^+ = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$.

What about its material time derivative?

$$\frac{\mathrm{D}\boldsymbol{\sigma}^+}{\mathrm{D}t} = \frac{\mathrm{D}\boldsymbol{Q}}{\mathrm{D}t}\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\frac{\mathrm{D}\boldsymbol{\sigma}}{\mathrm{D}t}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\frac{\mathrm{D}\boldsymbol{Q}^T}{\mathrm{D}t}.$$

Clearly the material time rate of the Cauchy stress tensor is not objective.

Objective stress rates

Starting from the material time rate of the Cauchy stress

$$\dot{\boldsymbol{\sigma}}^{+} = \dot{\boldsymbol{Q}}\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\dot{\boldsymbol{Q}}^T$$

and taking into account that

$$\boldsymbol{w}^{+} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{w}\boldsymbol{Q}^T \quad \Rightarrow \quad \dot{\boldsymbol{Q}} = \boldsymbol{w}^{+}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{w}.$$

Substituting it back

$$\begin{aligned}
\dot{\boldsymbol{\sigma}}^{+} &= \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + (\boldsymbol{w}^{+}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{w})\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}(\boldsymbol{w}^{+}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{w})^T \\
&= \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{w}^{+}\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{w}\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T(\boldsymbol{w}^{+})^T - \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{w}^T\boldsymbol{Q}^T \\
\dot{\boldsymbol{\sigma}}^{+} - \boldsymbol{w}^{+}\boldsymbol{\sigma}^{+} - \boldsymbol{\sigma}^{+}(\boldsymbol{w}^{+})^T &= \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{w}\boldsymbol{\sigma}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{w}^T\boldsymbol{Q}^T = \boldsymbol{Q}(\dot{\boldsymbol{\sigma}} - \boldsymbol{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{w}^T)\boldsymbol{Q}^T.
\end{aligned}$$

Define $\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{w}\boldsymbol{\sigma} - \sigma\boldsymbol{w}^T$ then

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{+} = \boldsymbol{Q}\,\overset{\circ}{\boldsymbol{\sigma}}\,\boldsymbol{Q}^T$$

is an objective rate of the Cauchy stress known as the **Jaumann-Zaremba** rate of the Cauchy stress. It is also called as co-rotational rate.

Objective stress rates (cont'd)

The Jaumann-Zaremba rate is very much used in large strain plasticity computations and many commercial FE programs use it in their implementation

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbb{C}^{\mathrm{e}} : (\boldsymbol{d} - \boldsymbol{d}^{\mathrm{p}})$$

However, it has some shortcomings which was observed by J.K. Dienes in 1979 (*Acta Mechanica*, Vol 32, pp. 217-232).

E.g. in simple shear $x_1 = X_1 + (t/t_0)X_2,\ x_2 = X_2,\ x_3 = X_3$ the solution for hypoelastic $\overset{\circ}{\boldsymbol{\sigma}} = \mathbb{C}^{\mathrm{e}} : \boldsymbol{d}$ produces oscillating solution

$$\begin{aligned}
\sigma_{12} &= G\sin(t/t_0), \\
\sigma_{11} &= G(1 - \cos(t/t_0)), \\
\sigma_{22} &= G(\cos(t/t_0) - 1),
\end{aligned}$$

where G is the shear modulus.

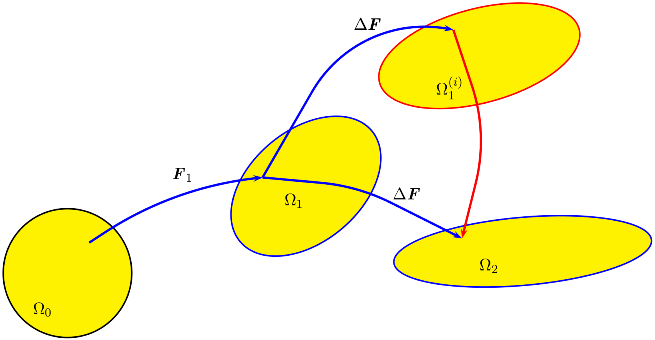
Objective stress rates (cont'd)

There are many other objective time rates, like (hear given as stress rates)

1. Oldroyd rate $\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^T.$
2. Cotter-Rivlin rate $\overset{\triangle}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{l}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{l}.$
3. Truesdell rate $\overset{*}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^T + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{d}.$
4. Green-McInnis-Naghdi $\overset{\square}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{R}}\boldsymbol{R}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\boldsymbol{R}}\boldsymbol{R}^T.$

Incremental descriptions

1. Total Lagrangian formulation. Reference configuration is the initial configuration Ω_0.
2. Updated Lagrangian formulation
   1. Reference configuration is the last equilibrium state Ω_1.
   2. Reference configuration is the state from the last iterate $\Omega_1^{(i)}$, weather or not it is in equilibrium.
3. Eulerian formulation. Reference to the current state Ω_2.



Principle of virtual work (PVW)

Total Lagrangian (TL) formulation

$$-\int_{\Omega_0} \delta \boldsymbol{E}_0 : \boldsymbol{S}_0 \, \mathrm{d}V_0 + \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V_0 + \int_{\partial\Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A_0 - \int_{\Omega_0} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_0 \, \mathrm{d}V_0 = 0$$

Updated Lagrangian (UL) formulation

$$-\int_{\Omega_1} \delta \boldsymbol{E}_1 : \boldsymbol{S}_1 \, \mathrm{d}V_1 + \int_{\Omega_1} \delta \boldsymbol{u} \cdot \rho_1 \bar{\boldsymbol{b}} \, \mathrm{d}V_1 + \int_{\partial\Omega_{t1}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A_1 - \int_{\Omega_1} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_1 \, \mathrm{d}V_1 = 0$$

Eulerian formulation

$$-\int_{\Omega_2} \delta \boldsymbol{e} : \boldsymbol{\sigma} \, \mathrm{d}V_2 + \int_{\Omega_2} \delta \boldsymbol{u} \cdot \rho_2 \bar{\boldsymbol{b}} \, \mathrm{d}V_2 + \int_{\partial\Omega_{t2}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A_2 - \int_{\Omega_2} \delta \boldsymbol{u} \cdot \ddot{\boldsymbol{u}} \rho_2 \, \mathrm{d}V_2 = 0$$

Variation or linearization of a spatial field is formally equivalent to the Lie time derivative.

Variation of the Almansi strain tensor

Variation of the Eulerian Almansi strain tensor:

1. Apply the pull back operation to obtain a material field.

$$\boldsymbol{F}^T \boldsymbol{e} \boldsymbol{F} = \boldsymbol{E}$$

2. Take the variation of the material Green-Lagrange tensor

$$\delta \boldsymbol{E} = \tfrac{1}{2}(\delta \boldsymbol{H}^T \boldsymbol{F} + \boldsymbol{F}^T \delta \boldsymbol{H}) = \mathrm{sym} \delta \boldsymbol{H}^T \boldsymbol{F}$$

3. Apply the push forward operation to obtain the spatial field:

$$\boldsymbol{F}^{-T} \delta \boldsymbol{E} \boldsymbol{F}^{-1} = \boldsymbol{F}^{-T} \tfrac{1}{2}(\delta \boldsymbol{H}^T \boldsymbol{F} + \boldsymbol{F}^T \delta \boldsymbol{H}) \boldsymbol{F}^{-1} = \boldsymbol{F}^{-T} \tfrac{1}{2}[(\mathrm{Grad} \delta \boldsymbol{u})^T \boldsymbol{F} + \boldsymbol{F}^T \mathrm{Grad} \delta \boldsymbol{u}] \boldsymbol{F}^{-1}$$

   Notice that the spatial gradient $\mathrm{grad} \delta \boldsymbol{u} = \mathrm{Grad} \delta \boldsymbol{u}\, \boldsymbol{F}^{-1}$, thus

$$\boldsymbol{F}^{-T} \tfrac{1}{2}[(\mathrm{Grad} \delta \boldsymbol{u})^T \boldsymbol{F} + \boldsymbol{F}^T \mathrm{Grad} \delta \boldsymbol{u}] \boldsymbol{F}^{-1} = \tfrac{1}{2}[(\mathrm{grad} \delta \boldsymbol{u})^T + \mathrm{grad} \delta \boldsymbol{u}].$$

Internal virtual work

It has to be equivalent

$$-\int_{\Omega_0} \delta \boldsymbol{E}_0 : \boldsymbol{S}_0 \, \mathrm{d}V_0 = -\int_{\Omega_2} \delta \boldsymbol{e} : \boldsymbol{\sigma} \, \mathrm{d}V_2$$

Taking into account equations

$$\boldsymbol{S}_0 = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} \quad \delta\boldsymbol{E}_0 = \boldsymbol{F}^T\delta\boldsymbol{e}\boldsymbol{F},$$

we get

$$-\int_{\Omega_0} \boldsymbol{F}^T\delta\boldsymbol{e}\boldsymbol{F} : \boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}J\mathrm{d}V_0 = -\int_{\Omega_2} \delta\boldsymbol{e} : \boldsymbol{\sigma}\mathrm{d}V_2.$$

Internal virtual work (cont'd)

Let us look a little bit closer the term $\boldsymbol{F}^T\delta\boldsymbol{e}\boldsymbol{F} : \boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$. It is easy to simplify in the index form

$$\delta E_{KL} = F_{pK}\delta e_{pq}F_{qL}, \qquad S_{KL} = JF^{-1}_{Km}\sigma_{mn}F^{-1}_{Ln},$$

the inner product is then

$$\begin{aligned}\delta\boldsymbol{E} : \boldsymbol{S} &= \delta E_{KL}S_{KL} = JF_{pK}\delta e_{pq}F_{qL}F^{-1}_{Km}\sigma_{mn}F^{-1}_{Ln} = J\delta_{pm}\delta_{qn}\delta e_{pq}\sigma_{mn} \\ &= J\delta e_{mn}\sigma_{mn} = J\delta\boldsymbol{e} : \boldsymbol{\sigma}\end{aligned}$$

Linearization of the internal virtual work

In the total Lagrangian formulation

$$-\int_{\Omega_0} \delta \boldsymbol{E} : \boldsymbol{S} \, \mathrm{d}V \tag{1}$$

Assuming constitutive equation in the form $\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ and we are in the displaced state $\boldsymbol{u}_1$ and we try to solve the increment to obtain $\boldsymbol{u}_2 = \boldsymbol{u}_1 + \Delta\boldsymbol{u}$. At the configuration 1 stresses are denoted as $\boldsymbol{S}_1$ and then

$$\boldsymbol{S}_2 = \boldsymbol{S}_1 + \Delta\boldsymbol{S} = \boldsymbol{S}_1 + \mathbb{C}\Delta\boldsymbol{E},$$

substituting it and $\delta\boldsymbol{E}$, $\Delta\boldsymbol{E}$ and $\boldsymbol{F}_2 = \boldsymbol{F}_1 + \Delta\boldsymbol{F} = \boldsymbol{F}_1 + \Delta\boldsymbol{H}$ into the internal VW-expression (1) gives

$$-\int_{\Omega_0} \tfrac{1}{2}[\delta\boldsymbol{H}^T(\boldsymbol{F}_1+\Delta\boldsymbol{H})+(\boldsymbol{F}_1^T+\Delta\boldsymbol{H}^T)\delta\boldsymbol{H}] : (\boldsymbol{S}_1+\mathbb{C}\tfrac{1}{2}[\Delta\boldsymbol{H}^T(\boldsymbol{F}_1+\Delta\boldsymbol{H})+(\boldsymbol{F}_1^T+\Delta\boldsymbol{H})\Delta\boldsymbol{H}]) \, \mathrm{d}V \tag{2}$$

About programming

How to set up `IEN`, `ID` and `LM` arrays.

- IEN(L,E) = global node number of local node L of an element E.
- ID(I,N) = global DOF number of local DOF I at global node N.
- LM(J) = Location Matrix, gives the global DOF of a local node J for element E.

LM array is redundant, it is not necessarily needed, it can be constructed from IEN and ID.

Next

Lecture.
Linearization of the internal virtual work + 1,2,3 D truss element.

Exercises on Thursday.
Numerical integration, code structure for element and internal force vector computations, quadratic isoparametric bar element.