FEM advanced course

Lecture 3 - Kinematics, time rates, elastic constitutive models

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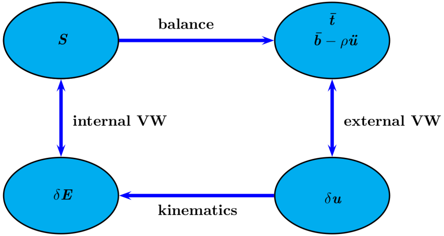
Principle of virtual work (PVW)



$$-\int_{\Omega_0} \delta\boldsymbol{E} : \boldsymbol{S}\,\mathrm{d}V + \int_{\Omega_0} \delta\boldsymbol{u}\cdot\rho_0\bar{\boldsymbol{b}}\,\mathrm{d}V + \int_{\partial\Omega_{t0}} \delta\boldsymbol{u}\cdot\bar{\boldsymbol{t}}\,\mathrm{d}A - \int_{\Omega_0} \delta\boldsymbol{u}\cdot\ddot{\boldsymbol{u}}\rho_0\,\mathrm{d}V = 0$$

$B^*\boldsymbol{S} = \rho_0\bar{\boldsymbol{b}}$ equilibrium

$\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ constitutive model

$\boldsymbol{E} = G\boldsymbol{u} \quad \Rightarrow \quad \delta\boldsymbol{E} = B\delta\boldsymbol{u}$ kinematical relation

Notice that the PVW is **independent of the constitutive model**.

Almansi strain tensor

Length of a line element PQ is $\mathrm{d}S = \sqrt{\mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X}}$,
In deformed state $|pq| = \mathrm{d}s = \sqrt{\mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}}$ notice that $\mathrm{d}\boldsymbol{X} = \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}$

$$\begin{aligned} \tfrac{1}{2}[(\mathrm{d}s)^2 - (\mathrm{d}S)^2] &= \tfrac{1}{2}(\mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X}) \\ &= \tfrac{1}{2}\mathrm{d}\boldsymbol{x} \cdot (\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1})\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{e}\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

where

$$\boldsymbol{e} = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{b}^{-1})$$

is the **Almansi strain tensor** and $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ is the **left Cauchy-Green deformation tensor**.

Almansi strain tensor is of Eulerian type. In dyadic form it can be written as

$$\boldsymbol{e} = e_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j,$$

where $\boldsymbol{e}_i$ are the unit base vectors of the spatial description.
(The Green-Lagrange strain tensor is expressed in the material description $\boldsymbol{E} = E_{IJ}\boldsymbol{E}_I \otimes \boldsymbol{E}_J$, where $\boldsymbol{E}_I$ are the unit base vectors in the material description.)

Some transformation formulas

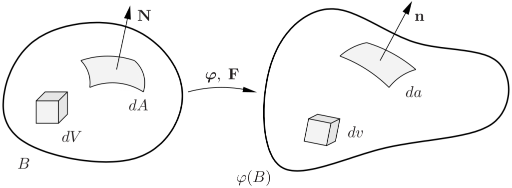
Area change between current and reference configuration

$$\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\mathrm{d}a = J\boldsymbol{F}^{-T}\boldsymbol{N}\mathrm{d}A = J\boldsymbol{F}^{-T}\mathrm{d}\boldsymbol{A}$$

It is known as Nanson's formula.

Volume change between current and reference configuration

$$\mathrm{d}v = J\mathrm{d}V, \quad J(\boldsymbol{X}, t) = \det \boldsymbol{F}(\boldsymbol{X}, t).$$



Pull back and push forward operations

We have to distinguish covariant (contravariant basis) and contravariant tensors (covariant basis).

Covariant tensors are often denoted as $\boldsymbol{E}^\flat$ and contravariant tensors as $\boldsymbol{\sigma}^\sharp$. Most strain/deformation tensors are covariant tensors, e.g. $\boldsymbol{E}^\flat, \boldsymbol{C}^\flat, \boldsymbol{e}^\flat, (\boldsymbol{b}^{-1})^\flat$. Contravariant deformation tensors are e.g. $(\boldsymbol{C}^{-1})^\sharp, \boldsymbol{b}^\sharp$.

Pull back operation (from spatial to material)

Covariant tensor $\boldsymbol{e}^\flat$:

$$\varphi_*^{-1}(\boldsymbol{e}) = \boldsymbol{F}^T \boldsymbol{e} \boldsymbol{F}$$

Contravariant tensor $\boldsymbol{\sigma}^\sharp$:

$$\varphi_*^{-1}(\boldsymbol{\sigma}) = \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}$$

Push forward operation (from material to spatial)

Covariant tensor $\boldsymbol{E}^\flat$:

$$\varphi_*(\boldsymbol{E}) = \boldsymbol{F}^{-T} \boldsymbol{E} \boldsymbol{F}^{-1}$$

Contravariant tensor $\boldsymbol{S}^\sharp$:

$$\varphi_*(\boldsymbol{S}) = \boldsymbol{F} \boldsymbol{S} \boldsymbol{F}^T \quad (= \boldsymbol{\tau} = J\boldsymbol{\sigma})$$

where $\boldsymbol{\tau}$ is the Kirchhoff stress.

Velocity gradient

Spatial velocity gradient $\boldsymbol{l}(\boldsymbol{x}, t)$ is defined as

$$\boldsymbol{l}(\boldsymbol{x}, t) = \frac{\partial \hat{\boldsymbol{v}}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} = \operatorname{grad} \hat{\boldsymbol{v}}(\boldsymbol{x}, t) \quad \text{or in index notation} \quad l_{ij} = \frac{\partial \hat{v}_i}{\partial x_j}.$$

Decomposing it into symmetric and antisymmetric (skew) parts

$$\boldsymbol{l}(\boldsymbol{x}, t) = \boldsymbol{d}(\boldsymbol{x}, t) + \boldsymbol{w}(\boldsymbol{x}, t)$$

where

$$\boldsymbol{d} = \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) = \boldsymbol{d}^T, \quad \text{and} \quad \boldsymbol{w} = \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^T) = -\boldsymbol{w}^T,$$

$\boldsymbol{d}$ is the **rate of deformation tensor** and $\boldsymbol{w}$ is the **spin tensor**.

Velocity gradient in terms of deformation gradient

$$
\begin{aligned}
\boldsymbol{l}(\boldsymbol{x},t) &= \operatorname{grad} \hat{\boldsymbol{v}}(\boldsymbol{x},t) = \frac{\partial \hat{\boldsymbol{v}}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} \\
&= \frac{\partial \dot{\boldsymbol{\varphi}}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \operatorname{Grad} \boldsymbol{v}(\boldsymbol{X},t) \boldsymbol{F}^{-1} = \frac{\partial}{\partial t} \left(\frac{\partial \boldsymbol{\varphi}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \right) \boldsymbol{F}^{-1} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}
\end{aligned}
$$

Material time derivative of a spatial feld

The material time derivative of a smooth spatial field $f(\boldsymbol{x}, t)$ is

$$\dot{f}(\boldsymbol{x}, t) = \frac{\mathrm{D} f(\boldsymbol{x}, t)}{\mathrm{D} t} = \left(\frac{\partial f(\boldsymbol{\varphi}(\boldsymbol{X}, t), t)}{\partial t} \right) \Big|_{\boldsymbol{X} = \text{constant}}$$

$$= \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial t} \Big|_{\boldsymbol{X} = \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t)} = \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \operatorname{grad} f \cdot \hat{\boldsymbol{v}}(\boldsymbol{x}, t)$$

The first term denotes the local time derivative of the spatial scalar field f, while the second term is called the **convective rate of change** of f, which is due to the change in position of particle $\boldsymbol{X}$.

Note that the material time derivative of a material field is just a normal time derivative, e.g.

$$\dot{\boldsymbol{E}}(\boldsymbol{X}, t) = \frac{\mathrm{D} \boldsymbol{E}}{D t} = \frac{\partial \boldsymbol{E}(\boldsymbol{X}, t)}{\partial t} = \ldots = \boldsymbol{F}^T \boldsymbol{d} \boldsymbol{F}.$$

Lee time derivative

Lee time derivative of a spatial tensor can be computed in the following way:

1. Apply the pull back operation to obtain a material field. As an example we consider the Lee time derivative of the Almansi strain tensor:

$$\boldsymbol{F}^T \boldsymbol{e} \boldsymbol{F} = \boldsymbol{E}$$

2. Take the material time derivative of the obtained material field:

$$\dot{\boldsymbol{E}}$$

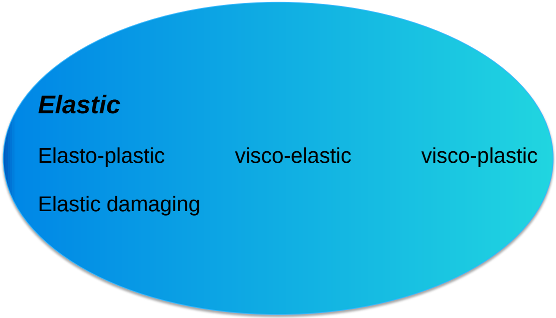
3. Apply the push forward operation to obtain the spatial field:

$$\boldsymbol{F}^{-T} \dot{\boldsymbol{E}} \boldsymbol{F}^{-1} = \boldsymbol{d}$$

Lee time derivative as presented here gives the time rate of change relative to the velocity field $\boldsymbol{v}$.

Constitutive models classification

Rate independent rate dependent



Symmetry classification

Eight possible linear elastic symmetries

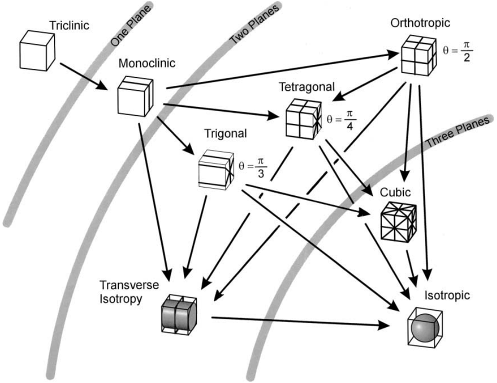


Figure from Chadwick, Vianello, Cowin, *JMPS*, 2001.

type of material symmetry	number of independent elastic coefficients
Triclinic	21
Monoclinic	13
Orthotropic	9
Tetragonal	6
Cubic	3
Trigonal	7
Transverse isotropy	5
Isotropy	2

Different types of elasticity

- Cauchy elasticity

$$\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\varepsilon}), \quad \text{or} \quad \boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}).$$

- Hypoelasticity

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{d}).$$

- **Hyperelasticity**

$$\boldsymbol{S} = 2\rho_0 \frac{\partial \psi(\boldsymbol{C})}{\partial \boldsymbol{C}}, \quad \text{or} \quad \boldsymbol{\sigma} = 2\rho \boldsymbol{b} \frac{\partial \psi(\boldsymbol{b})}{\partial \boldsymbol{b}},$$

  The constitutive equation is derived from a potential either from spesific Helmholtz free energy ψ. In isothermal problems it is equal to the spesific strain energy. In the following $\rho_0 \psi \equiv W$.

Isotropic elasticity

Isotropy means that the properties are the same in all directions.

The strain energy function can only be a function of the invariants

$$I_C, II_C, III_C, \quad \text{or} \quad I_b, II_b, III_b.$$

Representation theorem for isotropic elasticity: The most general form of isotropic elasticity is

$$\boldsymbol{\sigma} = a_0 \boldsymbol{I} + a_1 \boldsymbol{b} + a_2 \boldsymbol{b}^2,$$

where the coefficients a_0, a_1, a_2 can be non-linear functions of the **invariants**.

Notice that the invariants can be written in terms of the principal stretches

$$W(\boldsymbol{C}) \equiv W(\boldsymbol{b}) = W(\lambda_1^2, \lambda_2^2, \lambda_3^2)$$

Growth conditions to W:

$$\lim_{J \to +\infty} W = \infty \quad \text{and} \quad \lim_{J \to 0+} W = \infty.$$

Some examples of isotropic elastic models

Neo-Hooke for incompressible materials

$$W(I_C) = \frac{1}{2}\mu(I_C - 3).$$

Mooney-Rivlin (1940), (1948) model for incompressible materials

$$W(I_C, II_C) = c_1(I_C - 3) + c_2(II_C - 3).$$

Ogden (1972) model

$$W(\lambda_1, \lambda_2, \lambda_3) = g(J) + \sum_{i=1}^{r} \mu_i K_i(\lambda_1, \lambda_2, \lambda_3),$$

where

$$K_i(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{\alpha_i}\left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right)$$

Restrictions to parameters

$$\sum_{i=1}^{r} \mu_i \alpha_i = 2\mu, \quad \text{and} \quad \mu_i \alpha_i > 0.$$

Isotropic elastic models (cont'd)

One specific choice for the g-function, Ciarlet (1988):

$$g(J) = \frac{1}{4}\Lambda(J^2 - 1) - \left(\frac{1}{2}\Lambda + \mu\right)\ln(J),$$

and Λ, μ can be interpreted as Lamé constants.

Next

Exercises on Thursday at 2 PM in the CAD class K1242.

PVW in 1-D bar example using different constitutive model.

Next lecture, objectivity, updated Lagrangian formulation.