FEM advanced course

2. exercise – non-linear bar, kinematics, balance equation, linearization

1. Write the virtual work equation of a 1-D bar in the material description starting from the general form

$$-\int_{V_0} \delta \boldsymbol{E} : \boldsymbol{S} \, \mathrm{d}V + \int_{V_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V + \int_{A_{0\sigma}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A = 0, \tag{1}$$

   where $\boldsymbol{E}$ is the Green-Lagrange strain tensor, $\boldsymbol{S}$ the second Piola-Kirchhoff stress tensor, $\delta \boldsymbol{u}$ the virtual displacement field, $\bar{\boldsymbol{b}}$ the body force vector, $\bar{\boldsymbol{t}}$ the traction vector on the boundary part $A_{0\sigma}$ and ρ_0 is the density in the initial configuration.

   Initial length and cross-sectional area are L_0 and A_0, respectively. Perform the linearisation and interpret the result. Assume that $u(0) = u_1(0) = 0$ and a force H is acting at $X = X_1 = L_0$. The material is assumed to obey the St. Venant constitutive model

$$\boldsymbol{S} = \Lambda \mathrm{tr} \boldsymbol{E} \boldsymbol{I} + 2\mu \boldsymbol{E},$$

   where Λ, μ are the Lamé constans (see page 45 in the study book).

2. From the virtual work equation derive the equilibrium equation in strong form, i.e. differential equation form.

3. Solve the problem analytically assuming incompressible material. Draw the load-displacement curve.
   Hint. First take the deformation gradient $F = F_{11}$ as an unknown. Then in the second phase solve the displacement $u(X)$.

4. Formulate the problem 1 in the finite element setting: perform linearization, set up the $\boldsymbol{B}$ matrix, compute the stiffness matrix. Use linear interpolation functions and one element for the whole bar.

Home assignment 2. Code the non-linear bar finite element. This means program routines to set up

1. Interpolation functions, in this example use linear interpolation.

2. Use numerical integration. One point Gaussian quadrature is enough for element with linear interpolation.

3. Build routine to set up the $\boldsymbol{B}$-matrix and the internal force vector and the stiffness matrix (both material and geometric parts).

4. At this work you need not to code the assembly routine.

Design the routines in such a way that they can be easily extended to handle e.g. higher order interpolation and 2-3 dimensional problems. Solve the problem tackled in this exercise, rigid boundary at $X = 0$ and force H at $X = L$ using your code ($\bar{b} = 0$). In the analysis use one element with dimensions $L_0 = 100$ mm, $A_0 = 10$ mm^2, incompressible material with Young's modulus $\mathcal{E} = 0.1$ GPa. Compute to the maximum load 6 kN in tension and 0.18 kN in compression.

Solution report should be returned in Moodle prior to exercise 4