TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Architecture Institute of Building Climatology, Chair of Building Physics

Application of ADI Splitting Methods to Two-Dimensional Building Envelope System Solvers

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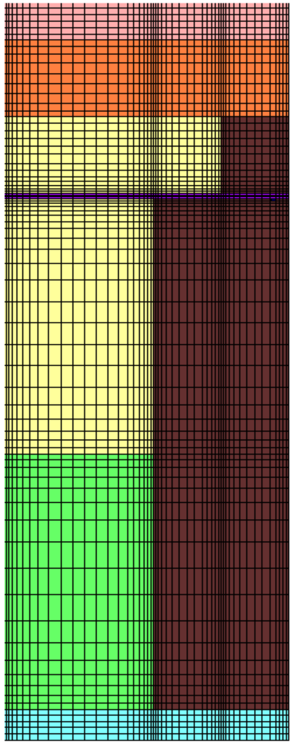
Motivation



Detail of a roof

Heat Transfer Equation

Thermal Balance

$$0 = c_T \rho \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_k} q_{cond,k}$$

$$q_{cond,k} = -\lambda \frac{\partial T}{\partial x_k}$$

Numerical Discretisation

- Space: Two-dimensional grid
- Time: Implicit or explicit time stepping method

ADI-Method

ADI Peaceman-Rachefort

$$c_T\rho\frac{T^{n+\frac{1}{2}}-T^n}{\frac{\Delta t}{2}} = -\frac{\partial}{\partial x}\left(-\lambda\frac{\partial T}{\partial x}\right)^{n+\frac{1}{2}} \quad -\frac{\partial}{\partial y}\left(-\lambda\frac{\partial T}{\partial y}\right)^{n}$$

$$c_T\rho\frac{T^{n+1}-T^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = -\frac{\partial}{\partial x}\left(-\lambda\frac{\partial T}{\partial x}\right)^{n+\frac{1}{2}} \quad -\frac{\partial}{\partial y}\left(-\lambda\frac{\partial T}{\partial y}\right)^{n+1}$$

Implicitly treated

Explicitly treated

ADI-Method

ADI Peaceman-Rachefort

$$c_T\rho\frac{T^{n+\frac{1}{2}}-T^n}{\frac{\Delta t}{2}} = \boxed{-\frac{\partial}{\partial x}\left(-\lambda\frac{\partial T}{\partial x}\right)^{n+\frac{1}{2}}} \quad -\frac{\partial}{\partial y}\left(-\lambda\frac{\partial T}{\partial y}\right)^{n}$$

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ADI-Method

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Implicitly treated
Explicitly treated

ADI-Method

Properties of classical ADI

- Implicit solution of a set of one-dimensional equations
- Tridiagonal Jacobian matrix
- Easy implementation in the case of regular grids
- Unconditional stable when applied to parabolic equations
- Loss of accuracy in presence of mixed-term-derivatives
- Limited suitability to parallelisation algorithms

ADI-Method

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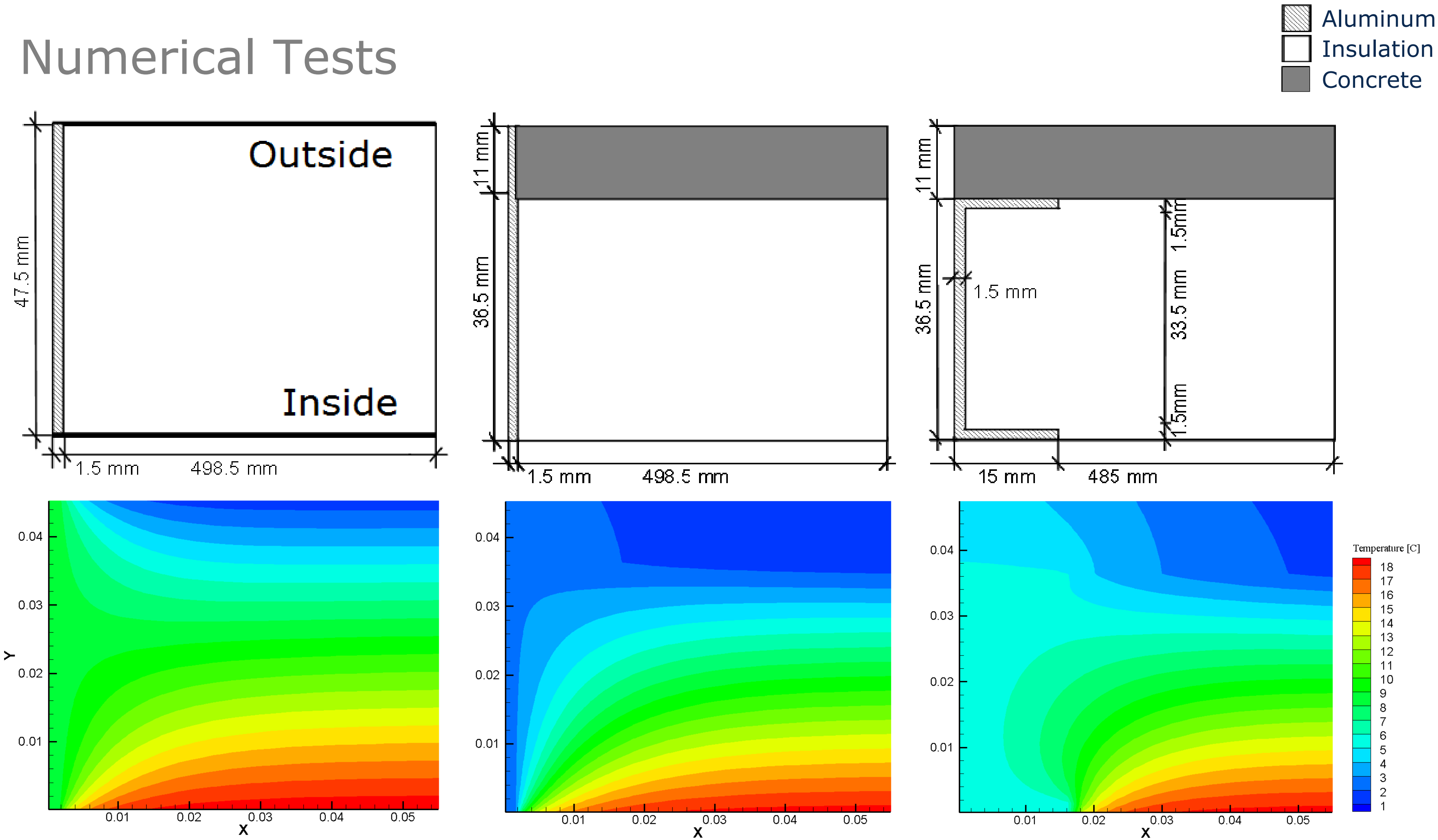
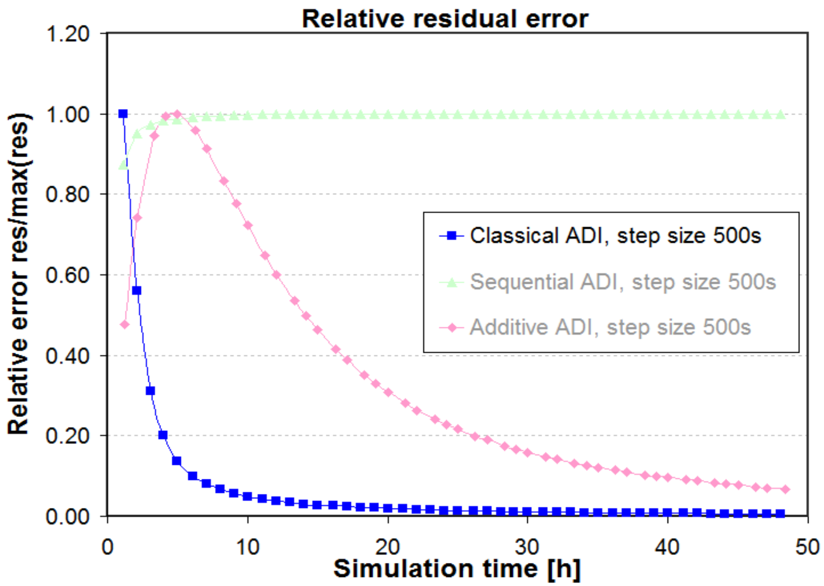
Numerical Tests



Numerical Tests



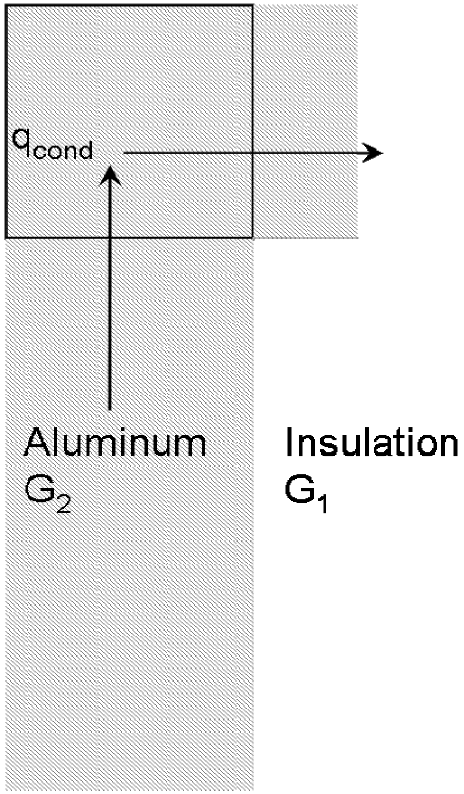
*Average time step size [s]: 0-10h/**10h-2d***

	Sequential	Classical	Additive
Case I	26/**27**	175/**252**	32/**95**
Case II	4/**5**	86/**189**	29/**79**
Case III	0.1/**0.1**	37/**75**	9/**23**

Results

- Test for accuracy: error estimated by comparison to implicit method
- Estimation of adaptive time step sizes
- Strict step size limitation for all cases
- Strong performance decrease with increasing geometrical complexity

Occurrence of Mixed-Term-Derivatives



Heat flux inside material G_i :

$$q_{cond,k}|_{G_i} = -\lambda_i(T)\frac{\partial T}{\partial x_k}$$

Helmholtz decomposition:

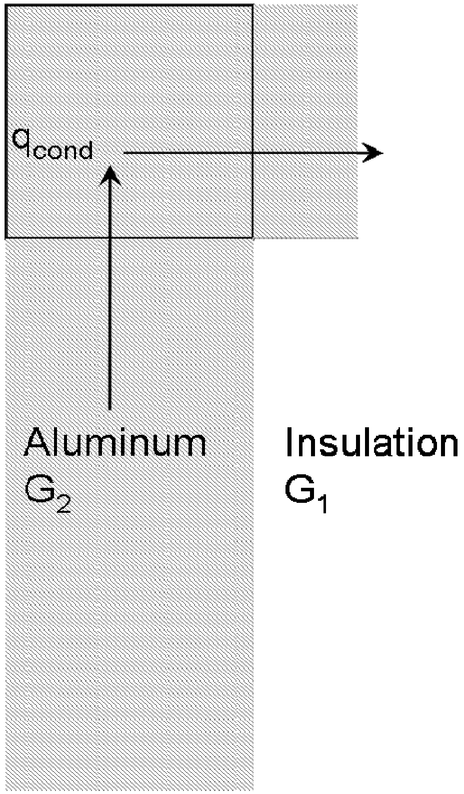
$$q_{cond} = \nabla\Phi + \nabla \times \Psi$$

Directional splitting:

$$\frac{\partial}{\partial x} q_{cond,1} = \frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial x \partial y}\Psi$$

$$\frac{\partial}{\partial y} q_{cond,2} = \frac{\partial^2}{\partial y^2}\Phi - \frac{\partial^2}{\partial x \partial y}\Psi$$

Occurrence of Mixed-Term-Derivatives



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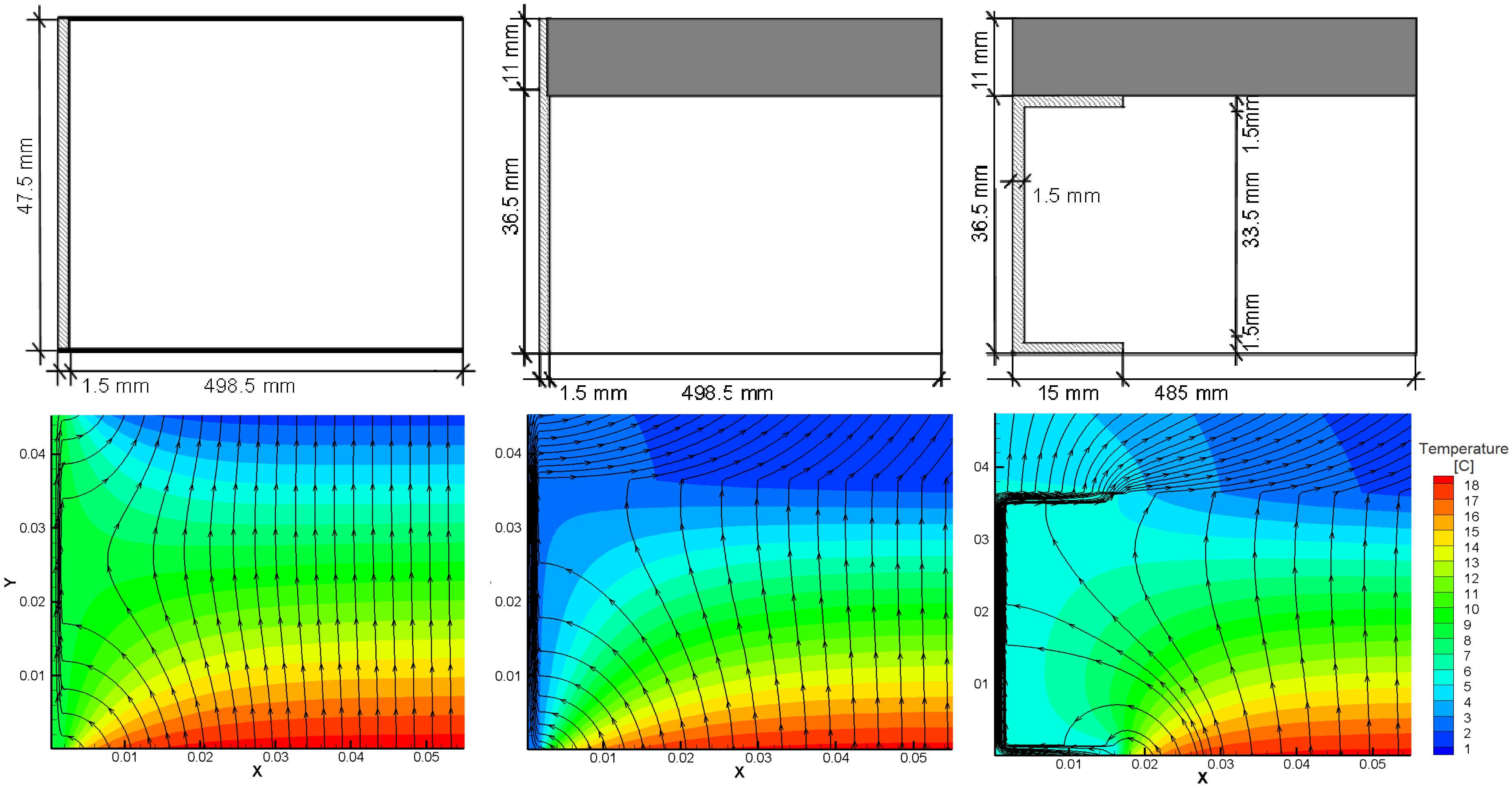
Occurrence of Mixed-Term-Derivatives

Aluminum
Insulation
Concrete



Conclusions

ADI-methods in BES simulation

- Acceptable accuracy requires small time step size
- ADI method is unsuitable for problems with discontinuous material properties, all balance types

Alternatives to direct application of ADI

- ADI achieves not an exact but a good approximate initial solution
- ADI can be easily transformed into a matrix preconditioner
- Alternative approach: Use of splitting preconditioning strategies combined with iterative linear equation system solvers

Thank you for your attention!

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