

PREFACE

Controversy has persisted in recent years over how to teach mathematical modelling. There are two extreme schools of thought. One believes that a modelling course should have little structure and be dominated by "case studies," open-ended exercises requiring data collection and research. These might lead students down a great many unforeseen avenues of creativity but might also lead them nowhere. The other school believes that a modelling course should be highly structured and dominated by what, for want of a better phrase, I will refer to as "modelling exercises." These are exercises that involve formulation and interpretation but bear a more certain relationship to the principal content of the course. Between these two extremes, of course, lie many compromises, one of which forms the basis for this textbook.

Before describing it, however, let me argue that there is an important sense in which the second school of thought should take precedence over the first. Most college majors in mathematics do not become professional mathematicians. Rather, they join the graduate workforce as businesspeople, as financial planners, as politicians, or in a number of other careers, and it is far more important that they can understand, interpret, criticize, and appreciate the mathematical models of others than that they are able to develop their own models. Just as drivers need not build and repair their cars but should know how they work, so consumers of mathematical models need not produce them but should understand both their usefulness and their limitations. Naturally, it remains desirable that math majors be able to develop their own models, but this is essential only to the few who subsequently make model development their career. On the other hand, the capacity to understand, interpret, criticize, and appreciate models is essential to all.

With this in mind, I have written this text for the deserving many, not simply a chosen few. Yet I have not neglected the specific needs of those who aspire to building their own models, although I have not explicitly addressed them until relatively late in the book. This is deliberate. Between the two extreme schools of thought this book has adopted a dynamic compromise. In which it begins by espousing the second extreme and gradually evolves toward the first. This approach enables the book to emphasize both the validation of mathematical models and the rationale behind improving them. Each of these features is novel, as will be clear to anyone familiar with Murthy and Rodin's (1987) review of books on mathematical modelling.

This is a book about the process of modelling. Its approach is heuristic but systematic and embodies my belief that the three most fundamental ideas in mathematical modelling are transience, permanence, and optimality. Because models incorporating these ideas may adopt either a deterministic or a probabilistic viewpoint, there are six combinations of idea and viewpoint. These define the structure of Chapters 1-3 and Chapters 5-7 of the text, which proceed in parallel as depicted in the following diagram.

	Idea	Transience	Permanence	Optimality
View				
Deterministic		Chapter 1 Growth and Decay- Dynamical Systems	Chapter 2 Equilibrium	Chapter 3 Optimal Control and Utility
Probabilistic		Chapter 5 Birth and Death. Probabilistic Systems	Chapter 6 Stationary Distributions	Chapter 7 Optimal Decision and Reward

In applying these ideas, mathematical modellers are called upon to play conflicting roles. On the one hand, they must be creative artists, assuming boldly perhaps what no one has assumed before. On the other hand, they must be critics, scrupulously doubting whether their models provide an adequate description of reality. In practice, of course, the distinctions I have drawn between ideas, roles, and even viewpoints are artificial; but it greatly benefits a beginning student of mathematical modelling to focus on a single aspect at a time, even though the other aspects cannot be entirely ignored. Accordingly, the creative aspect of modelling is the subject of Chapter 4, its critical aspect the subject of Chapter 9, and the interface between the two the subject of Chapter 8. The remainder of the book, Chapters 10-12, is a reinforcement of the ideas developed in the first nine chapters.

The goal of this book is to let readers acquire both critical and creative modelling skills and the confidence to use them. All other matters are regarded as secondary. Thus, as far as is possible, technical mathematical details are explored in exercises, or occasionally in footnotes, rather than in the main body of text. To distinguish such purely technical exercises from those that are more directly related to the goals of the book, modelling exercises are denoted by a single asterisk and case studies by a double asterisk. But not every detail has found its way into a footnote or exercise, and the book cannot be used profitably unless pencil and paper at all times accompany the reader. The book is to be studied, not simply read.

The book's goal is pursued through a layered approach, with frequent revisitations to earlier sections, so that even the most sophisticated models are perceived as merely natural outgrowths of less sophisticated ones. As proclaimed in the title, this layered approach is unashamedly concrete. Philosophical points are not discussed until several illustrative examples have already been introduced. Even then the discussion is brief, as at the beginning of Chapter 4, or in Chapter 9, where the art of adapting, extending, and combining is first discussed formally, even though it characterizes every model developed in the text. At all times, I have tried to avoid unnecessary abstractions. I have taken particular care never to introduce a utility function as $F(x)$, with constraints on the signs of the derivatives of F ; instead, I give the dependence of F on x explicitly, so that properties of the derivatives are self-evident. The loss of generality is more than compensated by the gain in comprehension. Indeed my experience suggests that the educational value of a modelling course to the average student is decided by this factor more than by any other.

This book is primarily intended for a senior level course that gives equal weight to deterministic and probabilistic modelling. Accordingly, minimal mathematical prerequisites for mastery of its entire contents are the standard calculus sequence (including the Newton-Raphson method) and first courses in linear algebra, ordinary differential equations, and probability and statistics. Probability and statistics are

reviewed in Appendix 1. A few sections require access to computer packages for solving linear programs or ordinary differential equations (now almost universally available), while some knowledge of numerical analysis is desirable (though by no means necessary). A modelling course need not include the book's entire contents, however, and alternative uses are described in an accompanying instructor's manual. In particular, there is ample material in Chapters 1-4 and Chapters 8-11 for a course that requires no probability and statistics.

But this book is designed to be also suitable for study alone, perhaps by beginning graduate students or by professionals whose background in mathematical modelling is weak. For this reason, if an exercise is crucial to later developments and of even moderate difficulty, then its solution (or a possible solution) appears at the back of the book. I therefore assume that my readers are mature enough not to consult a solution until a serious attempt has been made at the problem. Solutions to most other exercises appear in the instructor's manual.

By using plurals and genderless singulars, I have thus far avoided the vexed question of whether my reader is male or female. The lack of epicene pronouns and possessive adjectives in the English language is regrettable, and I have long advocated the use of "their" to mean "his or her." But grammarians may be offended by this, and the continual use of "his or her" is deplorably inelegant. I therefore felt that odd-numbered chapters should have one gender, even-numbered chapters another, and the toss of a coin decided that males would be odd, females even. Thus Chapters 1, 3, 5, 7, 9, and 11 are male, and Chapters 2, 4, 6, 8, 10, and 12 are female. This convention is not without minor inconveniences - for example, the forester we meet in Chapter 3 has to change his sex before she enters Chapter 4 - but this is surely preferable to the alternatives.

The reviews of the first edition of this book have all been very positive and pleasing. I haven't fixed what isn't broken. Nevertheless, for this revised edition I have corrected all known errors, I have added a few exercises, and I have updated both the list of references and other time-sensitive material.

Finally, this is a substantially original book. The models are based on the scientific literature, specifically, on the sources identified in Appendix 2. But I have made it a point of honor to personalize each model by deriving it from scratch, starting with a blank piece of paper and working through all the details myself. This has led not only to differences of emphasis and approach but also to new extensions and variations. Moreover, not only are models freely adapted, they are presented here in a fresh perspective, as the interlocking fragments of a mosaic. This is my mosaic of the modelling process, carefully assembled from my teaching experience, in which every fragment contributes to the synthesis of a coherent picture. What the picture shows me is that even a modest amount of mathematics - no more than should be expected of every college major in business, science, or engineering - is enough to describe a wide variety of phenomena, offer penetrating insights, and contribute effectively to rational decision making. If that is also what the picture shows my readers, then I shall be satisfied with what I have written.

Acknowledgement

I am deeply indebted to the many scientists whose mathematical models have formed a basis for this textbook. I wish to express my thanks to all of them, and to each in a measure proportional to my indebtedness.

I wish to thank Lou Gross for constructive and valuable criticism of the original manuscript, Steve Blumsack and Christopher Fye for discovering errors, and Charlie Nam for suggestions concerning the literature. I wish also to thank John Wiley & Sons for their support, cooperation, and professionalism in producing this revised edition.

Most of all, I would like to express my deepest gratitude to Karen, my wife, in whose space and time this book was written. Without her patience, encouragement and humor, it would never have been possible to complete the manuscript.

Tallahassee, Florida

MICHAEL MESTERTON-GIBBONS

CONTENTS

An ABC of modelling xix

I The Deterministic view

1 Growth and decay. Dynamical systems 3

1.1 Decay of pollution. Lake purification 5

1.2 Radioactive decay 7

1.3 Plant growth 7

1.4 A simple ecosystem 8

1.5 A second simple ecosystem 11

1.6 Economic growth 13

1.7 Metered growth (or decay) models 21

1.8 Salmon dynamics 23

1.9 A model of U.S. population growth 26

1.10 Chemical dynamics 29

1.11 More chemical dynamics 30

1.12 Rowing dynamics 32

1.13 Traffic dynamics 34

1.14 Dimensionality, scaling, and units 35

Exercises 40

2 Equilibrium 46

2.1 The equilibrium concentration of contaminant in a lake 52

2.2 Rowing in equilibrium 53

2.3 How fast do cars drive through a tunnel? 57

2.4 Salmon equilibrium and limit cycles 58

2.5 How much heat loss can double-glazing prevent? 63

2.6 Why are pipes circular? 66

2.7 Equilibrium shifts 71

2.8 How quickly must drivers react to preserve an equilibrium? 76

Exercises 83

3 Optimal control and utility 91

3.1 How fast should a bird fly when migrating? 93

3.2 How big a pay increase should a professor receive? 95

3.3 How many workers should industry employ? 103

3.4 When should a forest be cut? 104

3.5 How dense should traffic be in a tunnel? 109

3.6 How much pesticide should a crop grower use-and when? 111

3.7 How many boats in a fishing fleet should be operational? 115

Exercises 119

II Validating a Model

4 Validation: accept, improve, or reject 127

4.1 A model of U.S. population growth 127

4.2 Cleaning Lake Ontario 128

4.3 Plant growth 129

4.4 The speed of a boat 130

4.5 The extent of bird migration 132

4.6 The speed of cars in a tunnel 136

4.7 The stability of cars in a tunnel 138

4.8 The forest rotation time 142

4.9 Crop spraying 146

4.10 How right was Poiseuille? 148

4.11 Competing species 151

4.12 Predator-prey oscillations 154

4.13 Sockeye swings, paradigms, and complexity 157

4.14 Optimal fleet size and higher paradigms 159

4.15 On the advantages of flexibility in prescriptive models 161

Exercises 163

III Probabilistic View

5 Birth and death. Probabilistic dynamics 175

5.1 When will an old man die? The exponential distribution 180

5.2 When will N men die? A pure death process 183

5.3 Forming a queue. A pure birth process 185

5.4 How busy must a road be to require a pedestrian crossing control? 187

5.5 The rise and fall of the company executive 189

5.6 Discrete models of a day in the life of an elevator 193

5.7 Birds in a cage. A birth and death chain 198

5.8 Trees in a forest. An absorbing birth and death chain 200

Exercises 202

6 Stationary distributions 208

6.1 The certainty of death 210

6.2 Elevator stationarity. The stationary birth and death process 213

6.3 How long is the queue at the checkout? A first look 215

6.4 How long is the queue at the checkout? A second look 217

6.5 How long must someone wait at the checkout? Another view 219

6.6 The structure of the work force 225

6.7 When does a T-junction require a left-turn lane? 227

Exercises 234

7 Optimal decision and reward 237

7.1 How much should a buyer buy? A first look 237

7.2 How many roses for Valentine's Day? 243

7.3 How much should a buyer buy? A second look 245

7.4 How much should a retailer spend on advertising? 247

7.5 How much should a buyer buy? A third look 253

7.6 Why don't fast-food restaurants guarantee service times any more? 258

7.7 When should one barber employ another?

Comparing alternatives 263

7.8 On the subjectiveness of decision making 267

Exercises 268

IV The Art of Application

8 Using a model: choice and estimation 275

8.1 Protecting the cargo boat. A message in a bottle 276

8.2 Oil extraction. Choosing an optimal harvesting model 279

8.3 Models within models. Choosing a behavioral response function 281

8.4 Estimating parameters for fitted curves: an error control problem 285

8.5 Assigning probabilities: a brief overview 291

8.6 Empirical probability assignment 293

8.7 Choosing theoretical distributions and estimating their parameters 304

8.8 Choosing a utility function. Cautious attitudes to risk 316

Exercises 322

9 Building a model: adapting, extending, and combining 327

9.1 How many papers should a news vendor buy? An

adaptation 328

9.2 Which trees in a forest should be felled? A

combination 329

9.3 Cleaning Lake Ontario. An adaptation 334

9.4 Cleaning Lake Ontario. An extension 337

9.5 Pure diffusion of pollutants. A combination 345

9.6 Modelling a population's age structure. A first

attempt 350

9.7 Modelling a population's age structure. A second

attempt 360

Exercises 373

V Toward More Advanced Models

10 Further dynamical systems 383

10.1 How does a fetus get glucose from its mother? 383

10.2 A limit-cycle ecosystem model 389

10.3 Does increasing the money supply raise or lower

interest rates? 393

10.4 Linearizing time: The semi-Markov process. An

extension 398

10.5 A more general semi-Markov process. A further

extension 406

10.6 Who will govern Britain in the twenty-first century? A combination 409

Exercises 412

11 Further flow and diffusion 416

11.1 Unsteady heat conduction. An adaptation 417

11.2 How does traffic move after the train has gone by? A first look 421

11.3 How does traffic move after the train has gone by? A second look 423

11.4 Avoiding a crash at the other end. A combination 429

11.5 Spreading canal pollution. An adaptation 433

11.6 Flow and diffusion in a tube: a generic model 436

11.7 River cleaning. The Streeter-Phelps model 440

11.8 Why does a stopped organ pipe sound an octave lower than an open one? 446

Exercises 454

12 Further optimization 458

12.1 Finding an optimal policy by dynamic programming 458

12.2 The interviewer's dilemma. An optimal stopping problem 465

12.3 A faculty hiring model 470

12.4 The motorist's dilemma. Choosing the optimal parking space 475

12.5 How should a bird select worms? An adaptation 479

12.6 Where should an insect lay eggs? A combination 496

Exercises 507

Epilogue 514

Appendix 1: A review of probability and statistics 516

Appendix 2: Models, sources, and further reading
arranged by discipline 531

Solutions to selected exercises 539

References 583

Index 591