

Contents

Preface XIII

1 Introduction and General Outline 1

1.1 Mathematics and Modelling 1

1.1.1 Modelling Skills 4

1.1.2 Computation 5

1.2 Book Outline 5

I Mechanical Systems 9

2 Scales, approximations and solutions 11

2.1 Scaling and Inspectional Analysis 12

2.1.1 The flagpole 13

2.1.2 Scaling the flagpole problem 15

2.2 Approximations 20

2.2.1 The large J solution 20

2.2.2 The small J solution? 22

2.3 The Exact Solution 23

2.3.1 The small J solution 25

2.4 Dimensional Analysis 26

2.5 Summary 28

2.6 Exercises 29

3 A table fable 41

3.1 The Situation 41

3.1.1 Preliminary discussion 42

3.2 Background Information 44

3.3 The Simplest Model 46

3.3.1 The solution 50

3.3.2 Solution interpretation 52

3.4 Summary 54

3.5 Exercises 55

4 Moorings 63

4.1 The Situation 63

4.1.1 Preliminary discussion 64

4.2 Background Information 65

4.3 The Calculus of Variations 67

4.3.1 The archetype problem 67

4.3.2 Generalizations 73

4.4 Variational Principles in Dynamics 76

4.4.1 Hamilton's principle 80

4.4.2 Static continuous configurations 86

4.4.3 The hanging cable 87

4.5 A Cheaper Mooring 96

4.5.1 Setting up a model 96

4.5.2 The model 99

4.6 ****The Dynamics of a Moored Ship** 103

4.7 Summary 108

4.8 Exercises 108

5 The table fable revisited 121

5.1 Introduction 121

5.2 Energy Methods	122
5.3 The Supported Beam	127
5.3.1 The beam	128
5.3.2 Beam/posts matching	132
5.4 The Table	134
5.4.1 Plate solutions	135
5.4.2 Matching	138
5.5 Summary	140
5.6 Exercises	141
5.7 Appendix 1: The 6 legged table solution	142

II Diffusion 147

6 Preliminaries	149
6.1 Situations of Interest	149
6.2 Background	152
6.2.1 The experimental basis for heat flow studies	152
6.2.2 Mathematical background	156
6.3 The Heat Equation	156
6.3.1 Some elementary observations	158
6.3.2 Simple solutions of the heat equation	160
6.4 Mathematical Questions	163
6.4.1 Uniqueness theorems for the heat equation	164
6.4.2 The maximum principle	166
6.5 Summary	168
6.6 Exercises	168

7 Surface heating 177

7.1	Introduction	177
7.2	Mathematical Background	178
7.3	The Fundamental Plane Source Solution	180
7.3.1	Dimensional arguments	181
7.3.2	Invariant transformations	182
7.3.3	The similarity solution	184
7.4	Building Solutions	189
7.4.1	Infinite space problems	189
7.5	Other Fundamental Solutions	193
7.5.1	The method of images	194
7.6	Boundary Integral Methods	198
7.6.1	Moderate time behaviour	201
7.6.2	The asymptotic behaviour	204
7.6.3	Interpretation	205
7.6.4	General comments	205
7.6.5	The slab problem	206
7.7	Generalised Functions	208
7.8	Summary,	211
7.9	Exercises	212
8	Fourier methods	231
8.1	Introduction	231
8.2	Preliminaries	232
8.2.1	Sturm-Liouville theory	234
8.2.2	Basic Fourier series	236
8.3	Newtonian Slab Heating	239
8.4	Another Fundamental Slab Solution	243

8.5 Summary 246

8.6 Exercises 246

9 The art of cooking 253

9.1 Introduction 253

9.2 Surface Temperature Control 255

9.3 Temperature Distribution Control 258

9.4 Effects of Size and Shape 260

9.4.1 Special shapes 262

9.4.2 General shapes: engine blocks 266

9.4.3 The finite-element method 272

9.5 Summary 274

9.6 Exercises 275

10 Aspects of the greenhouse problem 283

10.1 Introduction 283

10.2 Temporal Fluctuations 285

10.2.1 Solution features 287

10.3 Spatial Variations 288

10.4 Adjusting to a new radiative equilibrium 290

10.4.1 Comments on the greenhouse effect 292

10.5 Summary 294

10.6 Exercises 295

11 Producing sheet steel? 299

11.1 Introduction 299

11.2 Solidifying The Steel 301

- 11.2.1 Heat transfer in a shallow moving medium 302
- 11.2.2 Solidification 303
- 11.2.3 The equations 303
- 11.2.4 The similarity solution 306
- 11.2.5 * Drum thickness effects 313
- 11.3 The revolving drum 316
- 11.4 Matching 321
- 11.5 Summary and Conclusions 322
- 11.6 Appendix: an alternative cooling system 322

III Vibrations and Waves 325

- 12 Vibrations 329
 - 12.1 Equilibrium and Vibrations 329
 - 12.2 The Linear Oscillator 331
 - 12.2.1 The forced undamped linear oscillator 332
 - 12.2.2 The damped linear oscillator 335
 - 12.3 Weakly Nonlinear Resonance 337
 - 12.3.1 The undamped nonlinear oscillator 338
 - 12.3.2 The damped nonlinear oscillator 343
 - 12.3.3 * Dissipation modelling 346
 - 12.4 Summary 349
 - 12.5 Exercises 349

13 Bumpy ride? 355

- 13.1 Introduction 355
- 13.2 The Situation 356
 - 13.2.1 Possible mechanisms 357

13.3 A Simplified Mathematical Model	359
13.4 The Resonant Behaviour	363
13.4.1 A preliminary analysis	364
13.4.2 Averaging	367
13.4.3 The amplitude and phase functions	371
13.5 Summary	376
13.6 Exercises	376
13.7 Appendix: Propshaft equations	379
14 Traffic flow	383
14.1 Introduction	383
14.1.1 Situations of particular interest	383
14.2 Steady Traffic Flow	384
14.2.1 Basic variables	384
14.2.2 The flux, speed, density relation	385
14.2.3 The flux vs density relation	386
14.2.4 A working flux vs. density relation	389
14.3 Modelling Unsteady Conditions	391
14.3.1 Conservation principle for cars	393
14.3.2 Solution construction	394
14.3.3 Starting from traffic lights	402
14.3.4 A traffic jam	404
14.4 Summary	411
14.5 Exercises	411
IV Solutions	423

Preface

Spurred on by past successes of mathematical modelling in the physical sciences, more recent successes in the biological sciences and the advent of cheap and powerful computing, the use of mathematical models has grown rapidly and spread to most disciplines. There is every reason to believe that this trend will continue to accelerate in the foreseeable future. The very recent growth of the use of models for examining problems arising in industry is especially marked. Industrialists have realized that mathematical models, when combined with computation, represent a far cheaper alternative to other options (experimentation etc.) for understanding many industrial processes. Basically, modelling and computing come cheap! It is hoped that this book will provide useful material for the training of students in the skills of mathematical modelling in general, and in industrial mathematics in particular. It's also hoped that the book will expose students to the challenges and fascination of mathematical modelling as a profession.

The attitudes and skills required for modelling are different from, but not incompatible with, those required for pure mathematics, and even differ from those normally emphasized in traditional applied mathematics courses. The authors believe that such attitudes and skills should be developed early in a student's background, and the only obvious way to do this is to expose students to real modelling problems. More specifically, students need to develop the ability to work with non-mathematical material presented in succinct form and to learn how to formulate models, determine solutions which are useful in context, and interpret the results. The need for such early modelling training has been generally recognized but, given the limited mathematical and contextual background of such students, and their limited exposure to the practical use of mathematics, it's not easy to design such courses. We've attempted to do this by carefully selecting the material (especially the models) and by using an algebraic package to provide technical assistance.

The book requires as mathematical background no more than second year calculus level (elementary differential equations, vector calculus etc.), together with the associated linear algebra, and assumes a small non-mathematical knowledge base. The non-mathematical background required is briefly discussed in the relevant chapters. The models examined relate largely to industrial and (to a lesser extent) scientific questions that the authors have encountered recently, and which are seen to be of current and future practical interest. Particular situations are presented (often as presented to the authors by the customer) and what follows is a discussion of the alternative approaches which may be adopted, followed by an analysis of the relevant models and an appraisal of the practical usefulness of the results obtained. Techniques are introduced when needed, and the advantages and disadvantages of such techniques are indicated in context. Errors in approach are also part of the learning experience and are presented. In a real sense an apprenticeship approach is taken.

Additionally the material has been selected so as to:

- Illustrate the use of mathematical techniques which are efficient practical, and frequently used.
- Reinforce significant ideas by meeting them in more than one context.

The cohesiveness necessary to do this places restrictions on content. For this reason the large scale discrete optimization problems and the statistical problems which often arise in industrial contexts were excluded. Other computer intensive models were excluded for the same reason.

-Present as much of the material normally regarded as essential for continuum modelling as possible. It is in the manner of presentation of this (and other) material that this book differs from traditional presentations.

-Draw the student's attention to the features that make different systems behave in similar ways. An awareness of such features is essential for modelling; this is why dynamics, diffusion and wave propagation are fundamental topics. Of course it's this commonality feature that enables modellers to contribute significantly over such a broad front.

One major difficulty encountered by students is that of handling straightforward but intricate and tedious technical calculations. This is a difficulty experienced by all mathematicians; however, with experience, one learns to avoid getting bogged down in the technical details. For the apprentice and professional alike symbol manipulation packages are a Godsend. At the press of a button Fourier coefficients to any order are calculated, differential equations solved, plots drawn etc. This enables one to direct one's mind to major issues such as the appropriateness of the solution procedure or the relevance of the calculations in the modelling context. It should be pointed out that such packages are not just a convenient tool for performing calculations - the methodology of mathematics is rapidly changing as a result of their introduction. Apart from the fact that such packages must be regarded as essential for present day mathematics usage, the use of Maple, just one of the competitive packages, in this book has enabled a realistic working approach to modelling to be adopted; students are stepped through real calculations. Thus using Maple has enabled the authors to present techniques that are commonly used in practice (rather than those restricted by class-time constraints) and to examine realistic models. Also, students are expected to carry out (close to) real size investigations, rather than the artificially simplified (often superficially related) calculations necessitated by time constraint in the pre-algebraic package era. It would be helpful if students had some familiarity with Maple but it's not essential; students are introduced slowly to the commands in the first few chapters and more commands are introduced when needed. The authors have in practice found it useful to present one lecture illustrating the capabilities of the package and refer students to notes or any of the many good books on Maple (eg. Ellis, Johnson, Lodi and Schwalbe (1992), or Heck (1993)). An up to date version of the Maple code used in the book will be maintained at the web site <http://maths.uwa.edu.au/~fowkes/homepage.html>, where worked solutions to exercises will also be available. Alternative algebraic packages can be used in association with the text, and Mathematica code for the book will also be maintained at the above site.

An unusual exercise format is adopted to facilitate the handling of these real problems. The context of the problem is explained and then a series of directives is given to assist the student's own investigations. Hints are provided to enable the student to discover a sensible approach, and the student is expected to interpret results in context, and is encouraged to experiment and suggest extension work. Initially this approach is somewhat daunting for students (for starters many of the exercises are rather longer than usual and more open ended); however, it has been found that confidence quickly develops. The examples form an essential component of the text. They either elaborate the text or examine analogous situations arising in often entirely different contexts. The connecting features are thus emphasized, and also it's hoped that students will develop the confidence to handle situations outside their normal experience and build up a useful knowledge base.

In order to facilitate the apprenticeship approach an informal conversational style is used. As we all know, mathematicians don't think the way they write. They write like lawyers - and no one can think like a lawyer writes! In writing the way they think the author hope that the intuitional model that enable them to see

through situations might also prove to be of value to students. At the very least it's hoped that the approach will encourage students to develop their own intuitive way of looking at situations.

The book is an expanded version of a course given mainly by the authors and Malcolm Hood to second year university students at the University of Western Australia on mathematical modelling. Malcolm's imprint on a number of the exercises will be recognized by students. The course evolved from a standard course on applied mathematics into a course based on various scientific investigations we have personally been involved in (of local and global origin) and later, as our interests spread, on industrial problems. Our interest in industrial problems was stimulated especially by the work of the Oxford Industrial Mathematics group, and the annual Australian Mathematics-in-Industry meetings run until this year by CSIRO.

A great deal of this book was completed during the last year of John's life when he was struggling with cancer; a very difficult period for him and his family. In spite of very limited energy resources John insisted on continuing because of his belief in the importance of such a work. The example of his courage is at least as valuable as the work contained in this book. The work would not have been possible if it wasn't for Jocelyn's (John's wife) extraordinary support; for this we're all grateful.

Many friends and colleagues have offered encouragement, provided welcome advice, corrected errors etc. I'd especially like to acknowledge the encouragement and advice from Bill Pritchard and Glyn Davies, and to thank Jenny Hopwood for critically commenting on the book material. Peter Chapman also offered useful advice on various chapters of the book, and Grant Keady offered Maple advice. I'm also grateful to my good friends Roman Bogoyev and Con Savas who were most generous with their time and help.

Working with John during the last year of his life was distressing, and putting the material together since his death has been most difficult. Without the help of dear friends (especially Paula) I could not have managed. All I can say is thanks!

Nev Fowkes

John Mahony

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